Ambiguity, Liquidity, and Price Efficiency∗

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Abstract
We develop a sequential trading model with ambiguity-averse market makers. Bid-ask spread contains a risk premium and an ambiguity premium. Higher ambiguity generally leads to higher market illiquidity, but there exist some thresholds at which liquidity suddenly dries up or floods, and the likelihood of occurrence depends on the salience of ambiguity relative to risk. Ambiguity impedes speed of learning and can generate “jump clustering”. Ambiguity skewness predicts future returns, hinders revelation of “tail risk” and can produce price over-reaction. Our model explains the market freezes and the “Flash Crash”, reconciles some conflicting empirical results, and provides new testable predictions.

Keywords: Ambiguity, Skewness, Market Liquidity, Return Predictability, Price Efficiency

JEL Classification: D81, G12, G14.

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1 Introduction

The steep drop in stock prices during the 2007-2009 financial crisis prompted many adversely afflicted countries to ban short-selling on stocks. For instance, in September 2008, the U.S. regulators unexpectedly adopted an emergency order that temporarily prohibited most short sales in financial stocks and subsequently issued a number of regulatory circulars. Even though several types of market makers were exempted from the ban as part of their market making and hedging activities, a great deal of ambiguity and confusion persisted in both the stock and equity options markets about the scope, duration, and implementation of the ban and about forthcoming unpredictable rulings. Battalio and Schultz (2011), Beber and Pagano (2013), Boehmer, Jones, and Zhang (2013) and others find that these shorting restrictions in the U.S. and other countries did little help to boost stock prices but were severely detrimental to stock and equity options market liquidity in terms of bid-ask spreads and other measures. In fact, the ambiguity among market makers resulted in increased spreads even for stocks and options that were unrestricted by the short-selling prohibition.

Similarly, the burst of the financial crisis was also accompanied by little or no trading of mortgage-backed securities (MBS) and collateralized debt obligations (CDO). The market freezes occurred despite the facts that no regulatory trading restrictions had been imposed on these assets and that market makers continued to post bid and ask prices for them (Easley and O’Hara (2010a)). Perhaps more surprisingly, the equity market went haywire and liquidity vanished during the “Flash Crash” of May 6, 2010 (CFTC and SEC (2010)). Easley, de Prado, and O’Hara (2012) suggest that the key reason was exit of market makers due to ambiguity about flow toxicity or probability of informed trading, while Cespa and Foucault (2014) use a feedback loop between asset liquidity, price informativeness, and dealers’ uncertainty across securities to explain the market meltdown.

Since the recognition of the distinction between risk and ambiguity (or equivalently, uncertainty) by Knight (1921) and Ellsberg (1961), it has been long believed that ambiguous information and investors’ ambiguity aversion play important roles in affecting portfolio choices, asset pricing, and market liquidity. A vast theoretical literature has been developed to investigate their impacts on the former two, yet, models of their effects on the

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1Boehmer, Jones, and Zhang (2013) and Battalio and Schultz (2011) estimate that the ban raised total trading costs in the U.S. stock and equity options markets by $600 million and $500 million respectively over the 14 trading days from September 18 to October 8, 2008.

2Caballero and Krishnamurthy (2008) provide several other famous historical episodes to show the impact of investors’ ambiguity on market liquidity.

3Knight (1921) first argues that individuals differentiate between risk (known probabilities) and uncertainty (unknown probabilities). Ellsberg (1961) popularizes the notions of ambiguity and ambiguity aversion by suggesting that most people prefer taking on risk in situations where they know specific odds as opposed to ambiguous ones.
latter are lacking. Two recent studies by Foucault, Pagano, and Röell (2013) and Vayanos and Wang (2013) aim at providing a comprehensive list of factors that determine market liquidity. Although uncertainty effect is emphasized repeatedly along the discussion of determinants such as participation, search, transaction costs, asymmetric information, imperfect competition, and funding constraints, investors’ ambiguity does not directly show up in any of the developed models. In contrast, besides the aforementioned empirical evidence, Chung and Chuwonganant (2014) show that the effect of VIX, a measure of market ambiguity, on stock liquidity is greater than the combined effects of all other common determinants of stock liquidity. In this paper, we tackle this challenge and build a sequential trading model à la Glosten and Milgrom (1985) that examines the determination and properties of market liquidity when ambiguity-averse market makers face (symmetric or asymmetric) ambiguous stock payoff and adverse selection from informed traders. Our model is also novel in introducing a measure of ambiguity skewness which indicates the relative salience of upside payoffs to downside payoffs. It deserves mentioning that even though the market makers are ambiguity averse, they update their priors dynamically according to the Bayes’ rule based on order flows.

Motivated by the famous Ellsberg paradox and related experimental evidence, existing studies exploring the implications of ambiguity on financial markets heavily rely on the maxmin expected utility (or multi-priors expected utility) axiomatized by Gilboa and Schmeidler (1989). In this framework, an ambiguity-averse investor evaluates an investment strategy according to the expected utility under the worst-case probability distribution in a set of prior distributions. In other words, she selects the strategy that maximizes the minimum expected utility over the set of possible distributions. In our model, we deviate from this preferences representation (where investors’ indifference curves are kinked) and instead assume that market makers’ preferences over wealth is represented by the smooth ambiguity expected utility. Klibanoff, Marinacci, and Mukerji (2005) conceptualize ambiguity as a mean-preserving spread in the distribution of expected utility induced by all possible probability measures and capture ambiguity aversion using lack of reduction between compound probabilities. Thus, an ambiguity-averse investor evaluates an investment strategy by comparing the expectation of a concave transformation of the expected utility of her final wealth conditional on each plausible payoff distribution. In addition to improved tractability, this alternative framework has several advantages. First, investors with smooth ambiguity utility exhibit more flexible behavior when compared to the extreme pessimistic ones gov-

Similarly, most existing theoretical work on short-sales constraints reviewed by Beber and Pagano (2013) focuses on their effect on security prices, but provides little connection between shorting restrictions and market liquidity.
erned by the maxmin utility, which helps to accommodate a variety of market phenomena. In fact, the smooth ambiguity utility nests the maxmin utility as a special case. Second, the smooth ambiguity utility makes an explicit separation between ambiguity and investors’ attitude toward ambiguity, while they are inseparable under the maxmin utility because the set of priors simultaneously determines both. This enables us to discuss whether and how ambiguity and ambiguity aversion are priced. We discover that they have different impact on market liquidity and returns. The separation between ambiguity and ambiguity aversion also enables us to discuss whether and how ambiguity plays a role in price efficiency while ambiguity aversion is fixed over time. We show that ambiguity affects price efficiency by influencing the speed at which the market digests information. Last but not the least, the smooth ambiguity utility allows to distinguish risk and ambiguity. The separation between risk and ambiguity enables us to separate ambiguity premium from risk premium and yield direct testable implications, and more importantly, to highlight the interaction between risk and ambiguity and its impact on liquidity. Furthermore, ambiguity and ambiguity skewness in our model enable us to address their joint impacts on liquidity, contemporaneous and future returns, and price efficiency.

Our main findings center on three aspects: market liquidity, return predictability, and price efficiency, and they can be summarized as follows: (1) Market liquidity. In general, higher ambiguity leads to higher market illiquidity, but there exist some thresholds at which market liquidity suddenly dries up or floods. Whether liquidity dry-up or flooding is more likely to happen depends on the salience of ambiguity relative to risk. (2) Return predictability. Ambiguity skewness on an asset’s payoff predicts the signs of its future return. More precisely, positive skewness predicts a negative future return, and vice versa. Importantly, how much ambiguity explains contemporaneous return and predicts future return depends on its skewness. For example, when the ambiguity is positively skewed, higher ambiguity leads to a higher contemporaneous return and a lower future return. (3) Price efficiency. Ambiguity impedes the speed of learning and can generates “jump clustering”. Ambiguity skewness hinders the revelation of extreme payoff or “tail risk” and may produce dynamic price over-reaction. Many of these findings are new to the theoretical literature. In particular, we provide a comparison between the smooth ambiguity utility and the maxmin utility and explain why these new findings are only obtainable under the former specification.

One of our contributions is to document and investigate the separation and interaction between ambiguity and risk in determining market liquidity. We derive the structural measures of risk premium and ambiguity premium in the bid-ask spread. Interestingly, we show that under symmetric ambiguity, the ambiguity premium is increasing in ambiguity but decreasing in risk aversion. This structural method sheds light on empirical asset pricing models that
consider both risk factors and ambiguity factors. We also find that ambiguity and ambiguity aversion have the opposite effects on the bid-ask spread in the sense that while higher ambiguity leads to a higher spread, higher ambiguity aversion narrows the spread. The reason behind this seemingly counter-intuitive result is that more ambiguity-averse market makers under competition pressure can protect themselves from adverse selection by changing the bid-ask midpoint to a greater extent while setting a smaller spread.

More importantly, we investigate the relationships between the interaction of risk and ambiguity, abrupt changes in liquidity, and price efficiency. Our main findings provide a lens to see through the shorting ban event, the market freezes, and the "Flash Crash" mentioned above. The main results are also in line with Ellul and Panayides (2011) who discover that exogenous analyst coverage termination leads to a significant deterioration of liquidity and price discovery, as a consequence of higher ambiguity among market participants relative to informed traders.

Remarkably, the model highlights the role of ambiguity skewness played in asset pricing. Our key result that positive ambiguity skewness on an asset’s payoff predicts its negative future return is analogous to the celebrated findings of Brunnermeier, Gollier, and Parker (2007) and Barberis and Huang (2008) that a positively skewed asset can be “overpriced” and earn a negative future return. In our model the ambiguity-averse market makers set a higher price to circumvent the “tail risk” that the positively skewed upside payoff is realized, whereas in theirs the driving forces are investors’ optimal expectations and distorted probability weighting, respectively. An emerging empirical literature documents that some asset skewness factors are priced in a way consistent with the theoretical predictions. Our paper confirms the importance of ambiguity skewness and presents testable implications on how it affects contemporaneous return, predicts future return, and influences price efficiency. In particular, our model points out the potential sources for some conflicting empirical findings on the predictability of ambiguity on future returns.

The remainder of the paper proceeds as follows. Section 2 relates our work with the existing literature. Sections 3 presents the model setup, and Sections 4 and 5 demonstrates how ambiguity, ambiguity aversion, and ambiguity skewness determines market liquidity, return predictability in a static context and price efficiency in a dynamic setting, respectively. Section 6 compares our results with those under maximin utility. Section 7 discusses the robustness, policy implications, and concludes. All proofs are relegated to the Appendix. Solutions to equilibrium of the dynamic trading are provided in the Online Appendix.

2 Literature Review

When applied to portfolio choice, the maxmin utility is most well-known for explaining investors’ limited or selective stock market participation (Dow and Werlang (1992)). Built on this result, Cao, Wang, and Zhang (2005) relate it to diversification (conglomerate) discount and equity premium, and Illeditsch (2011) links it to portfolio inertia in risky assets and excess volatility. Easley and O’Hara (2009, 2010b) suggest various regulations can moderate the effects of ambiguity and promote investors’ participation and firms’ going-public decisions, respectively. Most of these models are in static contexts, whereas in our sequential trading model, the market makers, despite being ambiguity averse, always participate to provide liquidity.

The impact of ambiguity on asset pricing is extensively studied by dynamic maxmin models. For instance, Epstein and Wang (1994) point out the possibility of amplification and price indeterminacy when uncertainty prevails in a Lucas tree economy. Chen and Epstein (2002) first establish that an asset’s excess return can be expressed as the sum of a risk premium and an ambiguity premium. Epstein and Schneider (2008) show that signals with ambiguous precision induce investors to react more strongly to bad news than to good news, and contribute to premia for idiosyncratic volatility as well as negative skewness in returns. Anderson, Ghysels, and Juergens (2009) derive the impact of risk and ambiguity on excess returns and find stronger empirical evidence for an ambiguity-return trade-off than for the traditional risk-return trade-off. While we do not have the space to provide a thorough account of this literature, interested readers are referred to Epstein and Schneider (2010) and Guidolin and Rinaldi (2013) for excellent reviews of the broad connection between ambiguity and finance.6

In contrast, only a few papers investigate the interaction between ambiguity and liquidity.7 Caballero and Krishnamurthy (2008) show that an increase in ambiguity or decrease in aggregate liquidity can generate a socially inefficient flight to quality, which can be ameliorated by a lender of last resort facility. Caballero and Simsek (2013) demonstrate a feedback mechanism between complexity arising from bank’s ambiguity about the financial network of cross-exposures and illiquidity resulting from possible fire sales. Easley and O’Hara (2010a) apply Bewley (2002)’s criterion for decision making under uncertainty to account for the market freezes observed during the 2007-2009 financial crisis. Routledge and Zin (2009) solve a

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6 Both discuss the theoretical underpinnings of maxmin model, smooth ambiguity model, and multiplier utility and related models inspired by robust control (Hansen and Sargent (2001, 2007), Maccheronia, Mari- nacci, and Rustichini (2006)), and review their applications in portfolio choice and asset pricing.

7 Similarly, studies on how ambiguity affect corporate investment decisions are scant (Garlappi, Gi- ammarino, and Lazrak (2013)).
monopolistic derivative market maker’s dynamic optimal portfolio and consumption choices and show that his ambiguity aversion can limit his ability to hedge a position, which widens the bid-ask spread and reduces liquidity. Ozsoylev and Werner (2011) show in a static setting that when competitive market makers choose not to participate under ambiguity, market depth (the reciprocal of price sensitivities) decreases and illiquidity risk (the probability of market illiquidity) increases. Except Easley and O’Hara (2010a), these papers assume the maxmin utility. Our paper is closest to Routledge and Zin (2009) and Ozsoylev and Werner (2011) in considering ambiguity-averse market makers, but our findings differ from theirs in several important aspects. First, the positive relation between ambiguity and market illiquidity disappears in Routledge and Zin (2009) if the market makers are competitive, whereas it still holds in our model. Second, none of our findings rely on the non-participation of market makers, whereas it is indispensable in Ozsoylev and Werner (2011). Third, the difference in utility specification and market structure enables us to derive many new results on the relationship between ambiguity and liquidity, which are not attainable under the maxmin utility. Finally, some findings regarding the effects of ambiguity on return predictability and market efficiency are new to the literature.

Our paper joins a small but growing literature on the smooth ambiguity utility. For instance, Caskey (2009) shows that it can generate both stock market underreactions exemplified in post-earnings announcement drifts and price momentum, and overreactions to accounting accruals. Gollier (2011) provides conditions under which more ambiguity aversion would increase an investor’s demand for ambiguous assets. Ju and Miao (2012) present a calibrated Lucas-tree economy in which risk-free rate, equity premium and equity volatility, among other things, are reproduced to match historical data.

3 The Model Setup

We consider a $T$-period sequential trading model à la Glosten and Milgrom (1985) with unit mass of traders and competitive market makers. At date 0, the fundamental of a risky asset is $\mu$. Its fundamental payoff at date $T + 1$ is either $\mu + \sigma$ in the $UP$ state or $\mu - \sigma$ in the $DOWN$ state with equal probability.\footnote{Because risk is not the focus of this paper, we assume risk distribution is symmetric for simplicity. The model can be extended to accommodate asymmetric distribution of risk and generate richer interactions between ambiguity and risk in asset pricing and market liquidity.} The probabilities of each state given $\mu$ and the asset’s risk $\sigma$ are known to all market participants. It is also common knowledge that the economy is in either a $GOOD$ or a $BAD$ world. In the former, $\mu = A > 0$, and in the latter, $\mu = -B < 0$. We assume that true world ($GOOD$ or $BAD$) and true state ($UP$
or \textit{DOWN}) are independent. The risk-free rate is normalized to be zero. For expositional convenience, the risky asset is labeled as a stock, but it can be alternatively interpreted as a mortgage-backed security (MBS) or a collateralized debt obligation (CDO).

At each date \( t \in \{1, \cdots T\} \), only one trader arrives and she can trade either one or zero unit of the stock.\footnote{Most extensions to \cite{GlostenMilgrom1985} keep this assumption for simplicity. \cite*{EasleyOHara1987} and \cite*{OzsoylevTakayama2010} allow different trade sizes with and without introducing new information uncertainty, respectively.} There is a continuum of informed traders with mass \( \rho \), each of whom knows the true final payoff at date 0. These is also a continuum of liquidity traders with mass \( 1 - \rho \), each of whom trades just for liquidity reasons and is insensitive to price. If a liquidity trader arrives the market, she submits a buy, sell, or zero order with probability \( r, r, \) and \( 1 - \rho - 2r \) respectively. The market makers do not know whether the trader is informed. There is no ambiguity about the distributions governing trade arrival and trade size, and the probability distribution is independent of the world and the state. In any period, the history of transactions is public information.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{market-makers-beliefs.png}
\caption{Market Makers’ Beliefs. Given that \( qA - (1 - q)B = 0 \), ambiguity is represented by \( A \) (the value of fundamental in the \textit{GOOD} world), and ambiguity skewness is measured by \( q \) (the probability of the \textit{GOOD} world). The asset’s risk is measured by \( \sigma \), and the probability of \textit{UP} and \textit{DOWN} states is equal.}
\end{figure}

We only depart from \cite{GlostenMilgrom1985} by assuming that the market makers become ambiguity averse following an event accompanied by ambiguous information, as
illustrated by the shorting ban and the “Flash Crash” events in the introduction. While the informed traders know perfectly about the true world and the true state, the identical market makers know that the economy will go UP or DOWN with equal (objective) probability in each world, but they are ambiguous about the true world and homogeneously assign a (subjective) probability \( q \in (0, 1) \) to the GOOD world and \( 1 - q \) to the BAD world, respectively.\(^{10}\) The knowledge of the market makers is represented in Figure 1. Note that when \( A = -B \), the economy is in only one world and the market makers are ambiguity-free. In order to capture the notion “ambiguity skewness” considered in empirical studies, we assume, after a proper normalization of \( A \) and \( B \), that

\[
qA - (1 - q)B = 0, \tag{1}
\]

implying that the market makers’ expected values of the fundamental at \( t = 0 \) and \( t = T + 1 \) are zero. It turns out that assuming a non-zero expected value \( c \) (without the normalization) only affects the “magnitudes” of the bid-ask midpoint or spread, but not their “signs or directions”. Therefore, none of our propositions will be qualitatively altered.

Given condition (1), one can simply measure ambiguity by \( A \) for any given \( q \), with a larger \( A \) implying higher ambiguity. When \( A = B \) and \( q = 1/2 \), the market makers face symmetric ambiguity. We refer to \( A \neq -B \) and \( q \neq \frac{1}{2} \) as the case with asymmetric ambiguity. In this case, when \( A > B \) and \( q < 1/2 \), we say that ambiguity is positively skewed, whereas when \( A < B \) and \( q > 1/2 \), ambiguity is negatively skewed. One can thus measure ambiguity skewness by \( q \), or equivalently, the relative magnitude or salience of \( A \) versus \( B \).

By assumption, the ambiguity beliefs of the market makers are common knowledge.\(^ {11} \) Note that the condition (1) is only restricted on the initial belief \( q \). As trading unfolds, the market makers learn from the order flows and update \( q \) over time, it will converge to zero or one.

In the presence of ambiguity and risk, the market makers’ preferences are assumed to be governed by the following smooth ambiguity expected utility function axiomatized by Klibanoff, Marinacci, and Mukerji (2005):

\[
U = E[\phi(E[u(w)|risk])|ambiguity].
\]

Namely, the ex ante welfare of the market makers is measured by the (second order) expectation of a concave function \( \phi \) of the (first order) expected utility \( u \) of their final wealth \( w \) con-

\(^{10}\)Typically one runs point estimates for the parameters of asset return. For second moments, one can argue that the limit of infinitely fine sampling would remove all estimation risk (Maenhout (2004)). However, it is notoriously difficult to estimate first moments (See, e.g., Welch and Goyal (2008)). Based on this, we introduce ambiguity through the first moment of the returns.

\(^{11}\)The market makers’ ambiguity beliefs are affected by market events and sentiments, which we assume are observable by the informed traders.
ditional on each plausible distribution for the stock payoffs. Specifically, \( E[u(w)|\text{risk}] \) is the standard expected utility under risk given each ambiguous prior, and \( E[\phi(E[u(w)|\text{risk}])|\text{ambiguity}] \) is the expectation of a concave transformation of \( E[u(w)|\text{risk}] \) across all ambiguity priors. When \( \phi \) is linear, the market makers are ambiguity neutral and act according to the standard expected utility, and the representation of the ambiguous beliefs can be reduced to a single compound probability distribution. However, when \( \phi \) is concave, the market makers are ambiguity averse and they dislikes any mean-preserving spread of the conditional expected utility \( E[u(w)|\text{risk}] \).

From now on we denote \( U = E[\phi(E[u(w)])] \) for notational simplicity. In order to obtain analytical solution, we follow Caskey (2009) and Gollier (2011) and consider \( u(x) = -e^{-\gamma x} \), where \( \gamma > 0 \) measures the degree of absolute risk aversion, and \( \phi(y) = -\frac{1}{\alpha}(-y)^\alpha \), \( \alpha > 1 \) where \( 1 - \alpha \) measures the degree of relative ambiguity aversion.\(^{12}\) Both are continuous functions. We will see shortly that not only are ambiguity and ambiguity aversion explicitly separated, they have very different impacts on asset prices and market liquidity.

At each date \( t \), before receiving an incoming order, the competitive market makers set the bid-ask midpoint, \( M_t \), minimize the (half) bid-ask spread \( S_t \), and make zero expected profit. We assume that market makers break even on average over both buy and sell orders.\(^{13}\) Note that this assumption, following Zhu (2014), is slightly different from that of Glosten and Milgrom (1985),\(^ {14}\) but it enables us to keep the economic intuition unchanged, and directly solve bid-ask spread and bid-ask midpoint which are the variables of interest in our paper. The market makers quote the ask price \( a_t = M_t + S_t \), the bid price \( b_t = M_t - S_t \), and a higher \( S_t \) corresponds lower market liquidity. We define the equilibrium in this economy as follows.

**Definition 1** The Perfect Bayesian Nash equilibrium should satisfy the following three conditions:

1. **Optimization condition.** Given their beliefs, the market makers set the bid-ask midpoint and spread as the best response.

2. **Tie-break condition.** Given the bid and ask prices, the informed traders buy or sell one unit of asset if they can make a positive profit.

\(^{12}\)As pointed out by Gollier (2011), this exponential-power specification for \( (u, \phi) \) differs from the exponential–exponential specification in Taboga (2005), the power-power specification in Ju and Miao (2012), and the power–exponential specification in Collard, Mukerji, Sheppard, and Tallon (2011). None of these three alternative specifications can be solved analytically.

\(^{13}\)Two model setups can justify the assumption that market makers break even at each date: (1) market makers only live for one period, and (2) Even though the market makers are long living, the market reaches the Perfect Bayesian Equilibrium (PBE) where market makers earn expected zero profit at each date.

\(^{14}\)In Glosten and Milgrom (1985), market makers set a break-even ask price contingent on receiving a buy order, and a break-even bid price contingent on receiving a sell order.
Bayes’ rule. The market makers’ beliefs over time are updated in a Bayesian fashion.\(^{15}\)

4 Static Equilibrium Analysis at Date 1

In this subsection we first discuss the determination of bid-ask midpoint \(M_1\) and spread \(S_1\) at date 1 when trading begins. It turns out the magnitude or salience of ambiguity \(A\) relative to risk \(\sigma\) plays an important role. For better illustration, we therefore first focus on a special case when the ambiguity is sufficiently small and present how the ambiguity-averse market makers solve their maximization problem.

If ambiguity \(A\) is sufficiently small such that \(b_1^* \geq A - \sigma\) and \(a_1^* \leq -B + \sigma\), as shown in Figure 2(a), then the market makers’ expected utility under risk is

\[
E[u(w)] = -\frac{1}{2} \exp\left\{-\gamma \left[(\rho + r)(M_1 + S_1 - \mu - \sigma) + (r)(\mu + \sigma - M_1 + S_1)\right]\right\}
- \frac{1}{2} \exp\left\{-\gamma \left[(\rho + r)(\mu - \sigma - M_1 + S_1) + (r)(M_1 + S_1 - \mu + \sigma)\right]\right\}
= -\frac{1}{2} e^{\gamma \rho (A-M_1)} \left( e^\gamma e^{(A-M_1)} + e^{-\gamma (A-M_1)} \right).
\]

Specifically, the first line shows when the state is \(UP\), each market maker faces a buy order with probability \(\rho + r\) and earns \(a_1 - (\mu + \sigma)\) or a sell order with probability \(r\) and earns \((\mu + \sigma) - b_1\), respectively. The second line can be interpreted similarly for the \(DOWN\) state.

The market makers’ expected utility under both ambiguity and risk is

\[
U = E[u(w)]
= -\frac{1}{2} e^{\gamma \rho (A-M_1)} \left( e^\gamma e^{(A-M_1)} + e^{-\gamma (A-M_1)} \right)
- \left(1 - q\right) e^{\gamma \rho (B+M_1)} \left( e^\gamma e^{(B+M_1)} + e^{-\gamma (B+M_1)} \right)
= -\frac{1}{\alpha} e^{\gamma \rho (A-M_1)} \left[ q(e^\gamma e^{(A-M_1)} + e^{-\gamma (A-M_1)}) + (1 - q)(e^\gamma e^{(B+M_1)} + e^{-\gamma (B+M_1)}) \right].
\]

Because the market makers are in a perfectly competitive market and the risk-free rate is normalized to be zero, we impose the zero expected profit condition so that in equilibrium,

\[
U = -\frac{1}{\alpha}.
\]

\(^{15}\)Epstein and Schneider (2010) report that the ambiguity literature widely uses Bayes’ rule for belief updating.
The $M_1$ is set to minimize $S_1$. We let

$$f(M_1) = q(e^{\gamma\rho(A-M_1)} + e^{\gamma\rho(A-M_1)})^\alpha + (1-q)(e^{\gamma\rho(B+M_1)} + e^{\gamma\rho(B+M_1)})^\alpha$$

and impose its first-order derivative with respect to $M_1$ to be zero,

$$\frac{1}{\gamma\rho} \frac{\partial f}{\partial M_1} = q\alpha(e^{\gamma\rho(A-M_1)} + e^{\gamma\rho(A-M_1)})^{\alpha-1}(e^{-\gamma\rho(A-M_1)} - e^{\gamma\rho(A-M_1)})$$

$$+ (1-q)\alpha(e^{\gamma\rho(B+M_1)} + e^{\gamma\rho(B+M_1)})^{\alpha-1}(e^{\gamma\rho(B+M_1)} - e^{-\gamma\rho(B+M_1)}) = 0.$$  

It is straightforward to show that the second-order condition is satisfied globally. From equations (2) and (4) we can solve the equilibrium $M_1^*$ and $S_1^*$, when the ambiguity $A$ is sufficiently small to satisfy $b_1^* \geq A - \sigma$ and $a_1^* \leq -B + \sigma$. Proposition 1 examines how ambiguity and ambiguity aversion are priced in this special case.

**Proposition 1** (Ambiguity, ambiguity aversion, and bid-ask spread) For sufficiently small ambiguity, higher ambiguity or lower ambiguity aversion charges a higher spread, no matter what ambiguity skewness is. Namely, $\frac{\partial S_1^*}{\partial A} > 0$ and $\frac{\partial S_1^*}{\partial (1-\alpha)} \leq 0$, $\forall q$, and the equality holds only when $q = \frac{1}{2}$.

Figure 2 is a graphical interpretation of Proposition 1. For example, in Figure 2(a) the spread $S_1^*$ rises when ambiguity $A$ increases for any skewness $q$. This appears intuitive as the ambiguity-averse market makers charge a higher ambiguity premium in $S_1^*$ for a larger $A$. In contrast, it may look odd that higher ambiguity aversion $1 - \alpha$ leads to a lower $S_1^*$ when ambiguity is asymmetric ($q \neq 1/2$). Figure 2 reveals that a change in $A$ or $1 - \alpha$ affects both $S_1^*$ and $M_1^*$ because the market makers strike a balance between them in order to protect themselves from the adverse selection. It turns out that a separate discussion on how $A$ or $1 - \alpha$ influences $M_1^*$ (see Proposition 4) is needed for a full understanding of Proposition 1. Fortunately, when ambiguity is symmetric ($q = 1/2$), $S_1^*$ can be expressed in a closed-form and $M_1^*$ is fixed at zero, so that the message of Proposition 1 can be clearly conveyed in this special case. We will return to Proposition 1 for the asymmetric case after discussing Proposition 4.

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16 In Proposition 1, sufficiently small ambiguity means $A \leq A_d$, where $A_d$ is defined in Proposition 2.

17 The effect of ambiguity aversion in Figure 2(b) is less obvious, but the numerical values verify that $S_1^*$ is indeed decreasing in $1 - \alpha$.
Figure 2: **Bid-Ask Midpoint and Bid and Ask Prices.** It is assumed that \( qA - (1-q)B = 0 \). Bid-ask midpoint is plotted in solid black line, and ask and bid prices are in dotted red lines. Parameter values: \( \rho = 0.2, r = 0.3, \gamma = 3 \). In figure (a), \( \alpha = 5 \). Ambiguity \( A \) ranges from 0 to 4. A larger value of \( A \) implies higher ambiguity. In figure (b), \( A = 1 \) and \( \sigma = 3 \). Ambiguity aversion \( (1 - \alpha) \) ranges from -6 to 0. A larger value of \( 1 - \alpha \) implies higher ambiguity aversion.
4.1 When Ambiguity is Symmetric

It is useful to compare the equilibrium without ambiguity and the equilibrium with symmetric ambiguity. We solve the former by imposing $A = -B$ in equations (2) and (4),

$$M^\text{na}_1 = 0,$$
$$S^\text{na}_1 = \frac{\rho}{\rho + 2r} \sigma,$$  \hspace{1cm} (5)

where subscript $na$ denote “no ambiguity”. Equation (5) generates a well-accepted result that when ambiguity is absent, $S^\text{na}_1$, being increasing in informed trading probability $\rho$ and risk $\sigma$ but decreasing in liquidity trading probability $r$, reflects the adverse selection cost stemming only from risk.\(^{18}\) We call $\frac{\rho}{\rho + 2r} \sigma$ the risk premium of the adverse selection cost for providing market liquidity.\(^{19}\)

Similarly, we derive the latter by imposing $A = B$ and $q = 1/2$ in equations (2) and (4),

$$M^\text{sa}_1 = 0,$$
$$S^\text{sa}_1 = \frac{\rho}{\rho + 2r} \sigma + \frac{\ln(e^{\gamma A} + e^{-\gamma A}) - \ln 2}{(\rho + 2r) \gamma},$$  \hspace{1cm} (6)

where subscript $sa$ denote “symmetric ambiguity”. Equation (6) shows that when ambiguity is present, $S^\text{sa}_1$ contains an additional component $\frac{\ln(e^{\gamma A} + e^{-\gamma A}) - \ln 2}{(\rho + 2r) \gamma}$ that is also increasing in $\rho$ but decreasing in $r$.\(^{20}\) For the reason that this component is dependent on ambiguity $A$ but independent on risk $\sigma$, we call it the ambiguity premium of the adverse selection cost for liquidity provision. To the best of our knowledge, we are the first to show that in the presence of symmetric ambiguity, the bid-ask spread contains a risk premium and an ambiguity premium.\(^{21}\) Additionally, in the symmetric case we clearly have $\partial S^\text{sa}_1/\partial (1 - \alpha) = 0$, so ambiguity aversion plays no role in affecting the spread.\(^{22}\)

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\(^{18}\)If we assume $qA - (1 - q)B = c \neq 0$, we have $M^\text{na}_1 = c$ and $S^\text{na}_1 = \frac{\rho}{\rho + 2r} \sigma$.

\(^{19}\)Notice that risk aversion $\gamma$ does not enter $S^\text{na}_1$ in equation (5), because we have assumed that the payoffs under a given world follow a symmetric binomial distribution. If we instead assume that the $UP$ and $DOWN$ states occur with probabilities $\theta$ and $1 - \theta$, respectively, where $\theta \neq 1/2$, then $S^\text{na}_1$ depends on $\gamma$ even though a closed-form solution is not attainable (see ). Since risk aversion is not the focus of our analysis, we think the assumed symmetric binomial distribution simplifies the calculation, keeps the intuition unchanged, and is thereby acceptable.

\(^{20}\)If we assume $qA - (1 - q)B = c \neq 0$, we have $M^\text{sa}_1 = c$ and $S^\text{sa}_1 = \frac{\rho}{\rho + 2r} \sigma + \frac{\ln(e^{\gamma (A-c)} + e^{-\gamma (A-c)}) - \ln 2}{(\rho + 2r) \gamma}$.

\(^{21}\)Chen and Epstein (2002) first demonstrate in a continuous-time setting that, when the representative agent has the maxmin utility, an asset’s excess return can be expressed as the sum of a risk premium and an ambiguity premium.

\(^{22}\)As indicated by Proposition 1, $1 - \alpha$ affects $S^*_1$ when $q \neq 1/2$ and the explanation is provided in the discussion of Proposition 4.
The symmetric ambiguity premium

\[ S_{1^{sa}} - S_{1^{na}} = \frac{\ln(e^{\gamma \rho A} + e^{-\gamma \rho A}) - \ln 2}{(\rho + 2r)\gamma} \]  

is always positive and increasing in ambiguity \( A \). More interestingly, equation (7) reveals an ambiguity-risk interaction effect under symmetric ambiguity: A higher risk aversion \( \gamma \) yields a lower ambiguity premium. The intuition is that given the same level of ambiguity, higher risk aversion makes ambiguity relatively less important, thereby yielding a lower ambiguity premium.

In Proposition 1, we assume that ambiguity \( A \) is sufficiently small. We next show that as \( A \) increases, the spread \( S_1^{*} \) jumps up or down when \( A \) crosses some critical thresholds. To better demonstrate this important finding, we continue to focus on the symmetric ambiguity case and provide a sketch of proof and a detailed discussion to Proposition 2, one of the most important findings of this paper. This result can be extended to the asymmetric ambiguity case, which will be displayed in Proposition 5, after we examine the impacts of ambiguity skewness, ambiguity, and ambiguity aversion on the midpoint \( M_1^{*} \).

**Proposition 2** (Symmetric ambiguity and market liquidity) When ambiguity is symmetric, there exist one liquidity dry-up threshold \( A_d \) and one liquidity flooding threshold \( A_f \), such that the spread \( S_1^{*} \) jumps up at \( A_d \) and jumps down at \( A_f \), and \( A_d < \sigma < A_f \). The midpoint \( M_1^{*} \) is always equal to zero no matter how large \( A \) is.

**Sketch of Proof.** Given risk \( \sigma \), when ambiguity is small such that \( A \leq A_d \), as Figure 3(a) shows, the ask price is lower than the payoff in the UP state of the BAD world and the bid price is higher than the payoff in the DOWN state of the GOOD world. When ambiguity is in an interval such that \( A \in (A_d, A_f) \), as Figures 3(b) and 3(c) display, the ask price is higher and the bid price is lower than both the payoff in the UP state of the BAD world and the payoff in the DOWN state of the GOOD world. When ambiguity is so large that \( A \geq A_f \), as Figure 3(d) exhibits, the ask price is lower than the payoff in the DOWN state of the GOOD world and the bid price is higher than the payoff in the UP state of the BAD world. The pricing rule in the intervals \( A \leq A_d \) and \( A \geq A_f \) is different from that when \( A \in (A_d, A_f) \). The reason is that when \( A \leq A_d \) and \( A \geq A_f \), the informed traders with information of the four different payoffs will submit non-zero orders if they arrive the market, but when \( A \in (A_d, A_f) \), the informed traders with information of the two medium payoffs will submit zero orders. Therefore, the liquidity changes when ambiguity \( A \) crosses the above two thresholds. The full proof is given in Appendix A.

This key finding is summarized in Figure 4. The liquidity dry-up or flooding are the results of the ambiguity-risk interaction and the market makers and the informed traders’
Figure 3: **Increasing Symmetric Ambiguity.** Ambiguity rises in figures (a) through (d). The informed traders with information on the medium payoffs trade only when the market makers’ ambiguity $A$ is lower than $A_d$ or higher than $A_f$.

The intuition behind it is as follows: When ambiguity is sufficiently small, the bid-ask spread is increasing in ambiguity. When ambiguity is at the liquidity dry-up threshold $A_d$, the bid price is equal to the *DOWN* state of the *GOOD* world and the ask price is equal to the *UP* state of the *BAD* world, so the informed traders knowing (true state, true world) is $(UP, BAD)$ or $(DOWN, GOOD)$ will still trade on their information.

When ambiguity exceeds $A_d$ by $\varepsilon$, the bid-ask spread increases. The bid price is lower than the *DOWN* state of the *GOOD* world and the ask price is higher than the *UP* state of the *BAD* world. The informed traders knowing (true state, true world) is $(UP, BAD)$ or $(DOWN, GOOD)$ will not trade, but the informed traders knowing (true state, true

Note that the functions in our smooth ambiguity utility are both continuous.
Figure 4: Symmetric Ambiguity and Market Liquidity. The market liquidity (measured by the bid-ask spread and represented by the bold solid curves) is high for low level of ambiguity. It dries up when ambiguity rises above $A_d$ and floods when ambiguity rises above $A_f$. It can be proved that the left and middle spread curves, determined by equations (A7) and (A9) in the Appendix A respectively, are concave, and the right curve, determined by (A13), is a straight line. The bid-ask midpoint is always equal to zero under symmetric ambiguity.

Other explanations are: (1) wealth effects due to decreasing absolute risk aversion (Kyle and Xiong (2001)); (2) interactions between tighter financing constraints and market illiquidity (Gromb and Vayanos (2002) and Brunnermeier and Pedersen (2009)); (3) cross-asset learning between liquidity and price informativeness (Cespa and Foucault (2014)).
price is lower than the \textit{UP} state of the \textit{BAD} world and the ask price is higher than the \textit{DOWN} state of the \textit{GOOD} world. In this case, the informed traders knowing (true state, true world) is \((\text{UP}, \text{BAD})\) or \((\text{DOWN}, \text{GOOD})\) will not trade, therefore the adverse selection cost only stems from the extreme payoffs. However, when ambiguity reaches \(A_f\), the bid price is equal to the \textit{UP} state of the \textit{BAD} world and the ask price is equal to the \textit{DOWN} state of the \textit{GOOD} world. In this case, the informed traders knowing (true state, true world) is \((\text{UP}, \text{BAD})\) or \((\text{DOWN}, \text{GOOD})\) restart to trade again, and the adverse selection cost is lowered by the medium payoffs. Therefore, the competitive market makers decrease the spread as a response to the discontinuous decline in the adverse selection cost.

Under symmetric ambiguity, the constant risk premium \((5)\) in the spread \((6)\) is independent of ambiguity. Proposition 2 carries an important message that ambiguity is a necessary condition for a sudden change in market liquidity, but whether ambiguity leads to a liquidity dry-up or liquidity flooding depends on the relative salience of ambiguity to risk. We can understand it in two aspects. First, one observation of Proposition 2 is that both thresholds are increasing in risk, i.e., \(\partial A_d/\partial \sigma > 0\) and \(\partial A_f/\partial \sigma > 0\). If we measure the resistance to ambiguity-induced liquidity dry-up with \(A_d\), the resistance is higher as the risk is higher. Second, when risk is more salient than ambiguity \((A < \sigma)\), the ambiguity is more likely to be at \(A_d\) and liquidity dry-up is more likely to happen. To the contrary, when ambiguity is more salient than risk \((A > \sigma)\), the ambiguity is more likely to be at \(A_f\) and liquidity flooding is more likely to happen. These two observations, being consistent with the idea behind equation \((7)\), send the same message that how ambiguity is priced depends on the level of risk.\(^{25}\)

Three remarks deserve special attention: First, the shorting ban event and other historical flight-to-liquidity episodes show that liquidity dry-ups are more often observed near ambiguous events. These empirical observations, in conjunction with Proposition 2, may imply that in most of the time the value of ambiguity \(A\) is smaller than the value of risk \(\sigma\). Second, it is noteworthy that when ambiguity continues to rise above \(A_f\), market becomes more illiquid. Even when liquidity floods at \(A_f\), the bid-ask spread could be higher than when liquidity dries up at \(A_d\). It is unwise to infer from our model that the best response to greater illiquidity is to elevate market ambiguity. Last but not the least, we observe that ambiguous events in reality are often followed by a series of actions taken by corporations, medias and regulators that help to ameliorate the ambiguity among market participants. For instance, the bid-ask spreads in stock and equity options market were narrowed since the removal of shorting ban, and even a five-second trading halt on E-Mini contracts was believed

\(^{25}\)As Klibanoff, Marinacci, and Mukerji (2005) point out, the second-order uncertainty (ambiguity) is built on the first-order uncertainty (risk).
to mitigate market ambiguity during the “Flash Crash”. We therefore believe that once \( A \) crosses \( A_d \), it is more likely to fall back to a lower level rather than continue to rise up to \( A_f \). In other words, higher liquidity following a dry-up period is a result of lower ambiguity smaller than \( A_d \) instead of higher ambiguity larger than \( A_f \).

We have shown above that the informed traders may stop trading when the market makers’ ambiguity is above the liquidity dry-up threshold. If we interpret the risky asset as a MBS and a CDO, our finding can partially explain the collapse of trading volume and the lack of transactions in these markets during the recent financial crisis, even though market makers continued to post bid and ask prices for these assets and no regulatory trading restrictions had been imposed on them. It is worth mentioning that the economic mechanism behind the market freezes in our model is distinct from the Bewley’s unanimity criterion and inertia assumption considered in Easley and O’Hara (2010a) and other explanations proposed in the literature.\(^{26}\) By examining stocks that lost all analyst coverage due to market developments and new regulations, Ellul and Panayides (2011) find that these stocks’ bid-ask spreads increase by about 30 percent and their volume decreases by half relative to a control group, as a result of higher ambiguity after termination. They also show that coverage termination makes the price discovery process less efficient, consistent with our model’s implications demonstrated in the subsequent dynamic equilibrium analysis.

Before proceeding, we discuss an empirical implication of our finding. Proposition 1 reveals that for sufficiently small ambiguity, the bid-ask spread is increasing in ambiguity. Proposition 2 further demonstrates that for ambiguity \( A \) being in the intervals \([0, A_d] \), \((A_d, A_f)\), and \([A_f, +\infty)\), the bid-ask spread is increasing in ambiguity. But we should be careful when we do regression exercises on the impact of ambiguity on liquidity. Figure 4 helps display the potential problems of a linear regression. If our sample of ambiguity is within the intervals 1, 3, and 5, the coefficient of interest is unbiased. If our sample is within the interval 4, however, the coefficient is biased and we come up with a false conclusion that liquidity is increasing in ambiguity. Even though our sample is less likely to be in intervals 4 and 5 because of the remedy responses taken by corporations, medias and regulators, we may end up with a false conclusion if the sample is in interval 2, which gives overestimated relationship between illiquidity and ambiguity.

\(^{26}\)Other explanations are: (1) a market seizes up as a result of greater asymmetric information (Akerlof (1970)); (2) new transactions have an adverse impact on the value of inventories because of the mark-to-market accounting (Bond and Leitner (2010)); (3) the tension between creditors and shareholders makes a manager unwilling to sell assets at low prices that reflected the possibility of future fire sales (Diamond and Rajan (2011)).
4.2 When Ambiguity is Asymmetric

In the presence of asymmetric ambiguity, the midpoint \( M^*_1 \) is no longer equal to zero and the spread \( S^*_1 \) does not have a closed-form solution. After investigating how ambiguity and ambiguity aversion affect \( M^*_1 \) and \( S^*_1 \), we are able to relate asymmetric ambiguity to return predictability and market liquidity, the former of which has received intensive attention in recent empirical studies (e.g., Connolly, Stivers, and Sun (2005), Anderson, Ghysels, and Juergens (2009), Bali and Zhou (2013), and Andreou, Kagkadis, Maio, and Philip (2014)).

For expositional clarity, we again first consider the case with sufficiently small ambiguity, then we generalize the analysis to the case with higher ambiguity.\(^{27}\)

**Proposition 3** (Ambiguity skewness and bid-ask midpoint) For sufficiently small ambiguity, the midpoint is not always an unbiased estimator for the asset’s expected payoff at \( t = 0 \) or \( t = T + 1 \).\(^{28}\) When ambiguity is symmetric, \( M^*_1 \) is equal to the expected payoff; When ambiguity is positively (negatively) skewed, \( M^*_1 \) is higher (lower) than the expected payoff, respectively.

Proposition 3 is clearly shown in Figure 2. For example, when \( q = 0.25 \), \( M^*_1 \) is above zero, which is the asset’s expected payoff at \( t = 0 \) or \( t = T + 1 \) because of the model setup and condition (1), for any level of ambiguity or ambiguity aversion. From the perspective of the ambiguity-averse market makers, they “rationally” set \( M^*_1 \), even though it may be biased (optimistic or pessimistic) in the eyes of the informed traders. When ambiguity is symmetric, an “unbiased” midpoint yields the smallest spread simultaneously. However when ambiguity is asymmetric, they have incentive to deviate from the “unbiased” midpoint in order to protect themselves from the realization of a extreme payoff and at the same time to make the most liquid market. For example, when ambiguity is positively skewed \((q < 1/2)\), the GOOD world is more salient, so they set an “optimistic” or “overpriced” \( M^*_1 \) (but it is still lower than \((A - B)/2\) because the salient GOOD world is less likely to happen).

Intuitively, the market makers exhibit the incentive to circumvent the extreme payoff as they, being ambiguity averse, dislike this “tail risk”. By setting the bid-ask midpoint closer to the extreme payoff (which is the GOOD world with positively skewed ambiguity), they can weaken its impact.

If we define contemporaneous return as the change in bid-ask midpoint from \( t = 0 \) to \( t = 1 \), and future return as the difference between the expected payoff at \( t = T + 1 \) and the

\(^{27}\) In Propositions 3 and 1, sufficiently small ambiguity means \( \Delta \leq \Delta_* \), where \( \Delta_* \) is defined in Proposition 4.

\(^{28}\) We say an estimator is “unbiased” if it is equal to the expected payoff at \( t = 0 \) or \( t = T + 1 \), otherwise we say the estimator is “biased”. For example, if the estimator is higher than the expected payoff, we say it is “biased” and “optimistic”.

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current bid-ask midpoint at $t = 1$, respectively, then a direct consequence of Proposition 3 can be summarized below:

**Corollary 1 (Ambiguity skewness and return predictability)** For sufficiently small ambiguity, positive skewness on an asset’s payoff predicts a positive contemporaneous return but a negative future return, and vice versa.

Remarkably, this key result is analogous to the famous findings of Brunnermeier, Gollier, and Parker (2007) and Barberis and Huang (2008) that a positively skewed asset tends to be overpriced and have a lower subsequent return. In the former, an investor with an optimal expectations framework can improve his well-being by endogenously choosing to believe that a big payoff of a positively skewed asset is more likely to occur. He is thus willing to pay a high price for it and accept its lower future return. In the latter, an investor under the cumulative prospect theory exogenously overweight the tails of an asset’s payoff distribution and values a positively skewed asset highly. Consequently this asset earns a negative average return. Our ambiguity aversion channel is complementary to these mechanisms. Additionally, in our model, it is the liquidity providers that set the price, while in theirs, the price is determined by the liquidity demanders. Barberis and Huang (2008) present empirical applications of their finding, and Boyer, Mitton, and Vorkink (2010) and Conrad, Dittmar, and Ghysels (2013) confirm the theoretical predication in more carefully designed empirical investigations. Strictly speaking, the ambiguity skewness measure in our paper is slightly different from the skewed returns in these empirical papers. A different line of literature focuses on the predictability of ambiguity rather than ambiguity skewness on contemporaneous and future returns. We need the following result to interpret these empirical studies.

**Proposition 4 (Ambiguity, ambiguity aversion, and bid-ask midpoint)** For sufficiently small ambiguity, higher ambiguity pushes the midpoint to lean towards the more salient payoff, whereas higher ambiguity aversion pushes it closer to the expected payoff. Namely, when $q < \frac{1}{2}$, $\frac{\partial M_t^*}{\partial A} > 0$ and $\frac{\partial M_t^*}{\partial (1-\alpha)} < 0$; and when $q > \frac{1}{2}$, $\frac{\partial M_t^*}{\partial A} < 0$ and $\frac{\partial M_t^*}{\partial (1-\alpha)} > 0$.

Figure 2 is a graphical representation of Proposition 4. For example, when $q = 0.25$, $M_t^*$ is getting further away from zero as $A$ increases, while it is getting closer to zero as $1 - \alpha$ rises. Same patterns are observed for $q = 0.75$ although the sign of $M_t^*$ is the opposite.

On the one hand, Proposition 4 shows that higher ambiguity $A$ induces the market makers to set a “riskier” price, in the sense that $M_t^*$ is farther from the expected payoff.

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29It is worth pointing out that the key finding stated in Corollary 1 is not driven by our assumption that the condition (1) holds and the asset’s future payoff is zero. It is standard in the literature to assume that the initial price of an asset equals the mean value of its future payoff.
To understand it, the market makers take the following three elements into account when clearing the market: (1) the cost of the extreme payoff, (2) the cost of ambiguity premium and risk premium, and (3) the benefit from liquidity trading. Suppose asymmetric ambiguity increases but the market makers keep \( M_1^* \) unchanged.\(^{30}\) As a result, costs (1) and (2) increase, so in order to break even, they have to charge a higher \( S_1^* \) to raise benefit (3). Besides raising \( S_1^* \), they can also adjust \( M_1^* \). By moving \( M_1^* \) in the “riskier” direction, the market makers lower cost (1). To break even, they set a little smaller \( S_1^* \) to lower benefit (3). Overall, we see \( \partial S_1^*/\partial A > 0 \), which is stated in Proposition 1.

On the other hand, Proposition 4 also shows that higher ambiguity aversion \( 1 - \alpha \) induces the market makers to set a “safer” price, in the sense that \( M_1^* \) is closer to the expected payoff, therefore \( A \) and \( 1 - \alpha \) have opposite impacts on \( M_1^* \) when ambiguity is asymmetric. Similarly, Proposition 1 demonstrates that \( A \) and \( 1 - \alpha \) have opposite impacts on \( S_1^* \). The reason behind \( \partial S_1^*/\partial (1 - \alpha) < 0 \) for \( q \neq 1/2 \) is as follows: \( S_1^* \) increases with ambiguity aversion is only true when \( M_1^* \) is fixed. However, the market makers under competition pressure respond to a change in ambiguity aversion by adjusting both \( S_1^* \) and \( M_1^* \) and striking a balance between self-protection and liquidity provision. Through adjusting the latter to a greater extent, the former can actually become smaller. In fact, this seemingly counter-intuitive result is akin to the finding in Gollier (2011): An investor may demand more ambiguous assets when she becomes more ambiguity averse.

Notably, the pricing rule employed by the ambiguity-averse market makers helps to create more space to learn from what is more likely to happen. For example, when the ambiguity is positively skewed \( (q < 1/2) \), the market makers believe that the true world is more likely to be \( BAD \). By setting a positive midpoint, the bid price and the ask price lean closer to the payoff in the \( GOOD \) world but further from the payoff in the \( BAD \) world. This pricing rule in turn motivates the market makers to learn more from the \( BAD \) world, by giving more profit to informed traders if they trade and making them to submit non-zero orders with a higher probability. To sum up, the ambiguity-averse market makers tend to employ a pricing rule that in turn encourages them to learn from what is more likely to happen but to neglect the extreme payoff with a small probability. As a result, this pricing rule over time hinders the revelation of “tail risk”, and we will discuss this implication further in the subsequent dynamic equilibrium analysis.

Recent empirical studies generally examine the impact of ambiguity \( A \) on contemporaneous and future returns. Propositions 3 and 4 imply that the relationship is not clear-cut because it depends on ambiguity skewness \( q \). More precisely, when ambiguity is symmetric, the relationship is null; when ambiguity is positively skewed \( (q < 1/2) \), a higher \( A \) leads to

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\(^{30}\)Preceding analysis shows when ambiguity is symmetric, \( M_1^* \) is fixed at zero.
a higher (and positive) contemporaneous return but a lower (and negative) future return; when the ambiguity is negatively skewed ($q > 1/2$), however, a higher $A$ yields a lower (and negative) contemporary return and a higher (and positive) future return. This is perhaps why recent empirical studies cannot come with an agreement on the relationship between ambiguity and returns. For instance, Anderson, Ghysels, and Juergens (2009) document a positive relationship between a measure of uncertainty and expected excess returns. Andreou, Kagkadis, Maio, and Philip (2014) find a negative relationship between stock market ambiguity and future market returns. Connolly, Stivers, and Sun (2005) and Bali and Zhou (2013) conclude that the relationship between ambiguity and contemporary returns is positive. Our model suggests that these conflicting findings might be reconciled if a measure of ambiguity skewness is introduced in the empirical investigation.

In Propositions 1, 3, and 4, we concentrate on the case where ambiguity $A$ is sufficiently small. Proposition 2 establishes the relationship between market liquidity and symmetric ambiguity at the different levels. For completeness, Proposition 5 shows that under asymmetric ambiguity, the bid-ask spread and midpoint jump up or down when $A$ crosses some critical thresholds which depend on ambiguity skewness $q$. We represent this result in Figure 5 but omit the discussion to preserve space and avoid repetition. Clearly, the symmetric ambiguity considered in Proposition 2 is a limiting case of Proposition 5 when $q \rightarrow 1/2$.

**Proposition 5 (Asymmetric ambiguity and market liquidity)** When ambiguity is asymmetric, (1) for any $q \neq 1/2$, there exist two liquidity dry-up thresholds $A_{d1}$ and $A_{d2}$, and two liquidity flooding thresholds $A_{f1}$ and $A_{f2}$, where $A_{d1} < A_{f1} < \sigma < A_{d2} < A_{f2}$, such that liquidity dries up or floods at the thresholds, and (2) at least at the neighborhood of $q = 1/2$, we have $\frac{\partial A_{d1}}{\partial |q-1/2|} < 0$, $\frac{\partial A_{f1}}{\partial |q-1/2|} > 0$, $\frac{\partial A_{d2}}{\partial |q-1/2|} < 0$, and $\frac{\partial A_{f2}}{\partial |q-1/2|} > 0$.

Again, we want to stress that, after an ambiguous event takes place and market liquidity dries up, market participants including corporations, medias and regulators typically take actions (such as board announcements, management interviews, media investigation, and regulatory clarifications) to alleviate or resolve ambiguity. Our simple model does not take such responses into consideration, but we believe that once $A_{d1}$ is triggered, ambiguity tends to fall back to a lower level and market liquidity will be higher as a consequence.

Given the proof of Proposition 5, one can fix the ambiguity level and plot a figure with market illiquidity and bid-ask midpoint on the vertical axis and ambiguity aversion on the horizontal axis. We choose not to display this because, even though theoretically one can show how bid-ask spread, midpoint, and returns change with ambiguity aversion, empiricists

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31 Recall that ambiguity aversion does not affect the bid-ask spread and the midpoint when ambiguity is symmetric.
We consider $q > \frac{1}{2}$. Generally the market illiquidity (measured by the bid-ask spreads and
represented by the solid curves above the horizontal axis) is higher and the bid-ask midpoints
(represented by the solid-dashed curves below the horizontal axis) are farther from zero for
higher level of ambiguity. They together jump up at $A_{d1}$ and $A_{d2}$ and jump down at $A_{f1}$ and
$A_{f2}$. Note that the figure is for illustration purpose. The curves are not necessarily straight
lines or parallel to each other, and their concavity/convexity depends on the parameters.

Figure 5: Asymmetric Ambiguity, Market Liquidity and Bid-Ask Midpoint.
can hardly find a robust time-varying proxy for ambiguity aversion. In practice, empirical work typically assumes ambiguity aversion to be stable or fixed over time and concentrates on examining the effects of measurable change in ambiguity.\footnote{Commonly used proxies for ambiguity include disagreement among analysts’ forecasts (e.g. Diether, Malloy, and Scherbina (2002) and Anderson, Ghysels, and Juergens (2009)), stock turnover (Connolly, Stivers, and Sun (2005)), the economic policy uncertainty index (Baker, Bloom, and Davis (2013)), and the VIX (Chung and Chuwonganant (2014)), among others.}

As a summary, we want to draw readers’ attention that the results stated in Propositions 1, 3, and 4 and Corollary 1 still hold when ambiguity $A$ falls within the intervals $[0, A_{d1}]$, $(A_{d1}, A_{f1})$, $(A_{f1}, A_{d2})$, $(A_{d2}, A_{f2})$, and $(A_{f2}, +\infty)$, in other words, the condition “for sufficiently small ambiguity” is sufficient but not necessary for the main findings. But even though the monotonicity of bid-ask spread and bid-ask midpoint with respect to ambiguity or ambiguity aversion is the same in different intervals, there exist jumps or structural changes at the four thresholds. Therefore we emphasize again that special care should be taken when running regressions of bid-ask spread or midpoint on ambiguity. In particular, if in reality after rising above $A_{d1}$, ambiguity is more plausible to decline before it reaches $A_{f1}$, piecewise regression should capture the relationship between illiquidity and ambiguity more accurately than a single linear regression.

5 Dynamic Equilibrium Analysis at Date $t$

We turn to present the dynamic equilibrium analysis of the sequential trading model considered above. Online Appendix provides equilibrium conditions at date $t$, from which the bid-ask midpoint and spread can be solved numerically. The basic idea is that at each date the market makers break even on average over both buy and sell orders, and take into account (1) the beliefs on each state of each world, which are time-varying;\footnote{In the static analysis, we impose the restriction $qA - (1 - q)B = 0$ on the initial belief $q$. As the market makers learn from the order flows, they update $q_t$ over time and competitively set $a_t$ and $b_t$ accordingly. As $t \to \infty$, $q_t$ will converges to zero or one and $q_tA - (1 - q_t)B$ will converge to $A$ or $-B$.} (2) the knowledge about the likelihood of informed trading and liquidity trading, which is fixed over time; and (3) the beliefs on the upcoming orders, which depends on the bid and ask prices and thus are time-varying. In our numerical exercises, we set $T = 1000$ and simulate 500 times (cross sections). The plotted ask and bid prices, spread and midpoint are all cross-sectional averages.

Once we obtain the solutions, we can investigate the evolution of price and liquidity. In the following discussion, we consider sufficiently small ambiguity most of the time, but similar conclusion can be made for higher ambiguity.
Figure 6: Ambiguity and Speed of Information Incorporation. We assume $A = -B$ in figure (a) and set the true state is UP. We assume $A = B$ and $q = \frac{1}{2}$ in figure (b) and set the true state and true world are UP and GOOD, respectively. We set $T = 1000$ and simulate 500 times (cross sections). The bid-ask spread (dashed blue curve), bid-ask midpoint (solid black curve), and ask and bid prices (dotted red curves) are all cross-sectional averages. Parameter values: $\rho = 0.2$, $r = 0.3$, $\gamma = 3$, $\alpha = 5$, $\sigma = 10$, and $A = 1$. 
Remark 1 (Ambiguity and speed of information incorporation) Ambiguity lowers the speed of information incorporation.

This result is illustrated by comparing the benchmark equilibrium without ambiguity in Figure 6(a) with the equilibrium under symmetric ambiguity in Figure 6(b). Clearly, under ambiguity, the bid-ask spread converges to zero and the bid-ask midpoint converges to the true payoff but at a lower speed. That is, under the impact of ambiguity, ambiguity-averse market makers tend to learn more slowly. Intuitively, in the presence of ambiguity, the process of uncertainty resolution tends to be slower given the source of information, i.e., order flows, is unchanged.

The direct implication of Remark 1 is that when ambiguity is prevailing, the market learns slowly and stay illiquid for a long time and price informativeness declines. This finding is consistent with several historical episodes described in Caballero and Krishnamurthy (2008), the recent shorting ban events, the theoretical prediction of Cespa and Foucault (2014), and the empirical evidence in Ellul and Panayides (2011). On the contrary, it is widely observed that when ambiguity among market participants are mitigated or resolved, the once evaporated liquidity is injected into the market quickly and price discovery process speeds up.

Recall that Proposition 2 proves that liquidity jumps at the liquidity dry-up or flooding thresholds while the plots in Figure 6 seems to be quite smooth. This is because we have assumed small ambiguity and only report cross-sectional averages of prices (bid and ask prices, midpoint, spread) so that the jumps are smoothed out. If we only plot a single draw, then the following Remark 2 displays that when ambiguity is not too small or too large, price jumps occur as information flow comes in gradually.

Remark 2 (Ambiguity and dynamic price jumps) Moderate level of ambiguity generates price jumps as information flow arrives sequentially.

Figure 7 illustrates the dynamic price jumps under symmetric ambiguity. Figure 7(a) shows the case where ambiguity is sufficiently small (smaller than the liquidity dry-up threshold $A_d$), while Figure 7(b) is the case where ambiguity is between the two liquidity jump thresholds ($A \in (A_d, A_f)$). The prices discontinuously jump at some dates in Figure 7(b), by comparison, the plots in Figure 7(a) are relatively more “smooth”.

Figure 8 helps to explain these difference. Suppose the true payoff is in the $(UP, GOOD)$ state, and ambiguity is neither too small nor too big. At date 1, the market makers set the price as in Figure 8(a1). As information flow comes in sequentially, the market makers update their beliefs and adjust the prices towards the true payoff, and after some trading
Figure 7: **Ambiguity and Dynamic Price Jump.** We assume symmetric ambiguity ($q = 1/2$) and set the true payoff is in the $(UP, GOOD)$ state. We set $A = 1$ in figure (a) and $A = 9$ in figure (b). Let $T = 1000$. We simulate 500 times (cross sections) and report the result of the 10th cross section. The bid-ask spread is plotted in dashed blue line, bid-ask midpoint is in solid black line, and ask and bid prices are in dotted red lines. Parameter values: $\rho = 0.2$, $r = 0.3$, $\gamma = 3$, $\alpha = 5$, and $\sigma = 10$.
Figure 8: **Ambiguity and Learning** Suppose the true payoff is in the \((UP, GOOD)\) state. When ambiguity is larger (between the two liquidity jump thresholds in Proposition 2), the pricing rule evolves from figures (a1) to (b1). In contrast, when ambiguity is sufficiently small (smaller than the liquidity dry-up threshold in Proposition 2), the pricing rule of the market makers evolves from figures (a2) to (b2), as they learn from the order flows gradually.

rounds, they set the prices as in Figure 8(b1). However, during the process from Figure 8(a1) to Figure 8(b1), because \(-B + \sigma\) is close to \(A - \sigma\), the liquidity dry-up threshold \(A_d\) in Proposition 2 is crossed at some point when the market makers are not sure enough about the true payoff. As a result, the prices jump to a larger extent at this threshold. In contrast, if ambiguity is sufficiently small, the prices evolve from Figure 8(a2) to Figure 8(b2). Because \(-B + \sigma\) is farther from \(A - \sigma\), when \(A_d\) is crossed, the market makers have learned more from the order flows and become better informed about the true payoff. Therefore, the change of price in the latter case is more gradually and the jump, if any, is less significant.

Two empirical implications of Remark 2 are in order. First, at most of the time when ambiguity is smaller than risk, the market reacts more drastically to given news under higher ambiguity. Even though the fundamental does not change, the ambiguity-averse market makers learn over time and “rationally” set more volatile prices. This implication matches our observation during crisis that ambiguity and market volatility go hand in hand. It is worthy
to point out that the drastic price changes (even under the \((UP, GOOD)\) state) plotted in Figure 7 are reminiscent of the radical price drop and rebound during the “Flash Crash” event of May 6, 2010 (CFTC and SEC (2010)). While Easley, de Prado, and O’Hara (2012) and Cespa and Foucault (2014) briefly touch on the role of uncertainty, we explicitly model it to interpret the market meltdown. There is another important difference between our model and Cespa and Foucault (2014). We only consider one asset and liquidity crash occurs after ambiguity rises and crosses the liquidity dry-up threshold. They model dealers’ cross-asset learning when assets have two-factor structure, therefore, a small exogenous increase in the illiquidity of one asset can lower price informativeness and raise uncertainty faced by the dealers, thus trigger a large drop in the liquidity of other assets.

Second, ambiguity can help to explain the “jump clustering” hypothesis in the asset pricing literature, which discovers that jumps are more likely to occur following the previous jump.\(^{34}\) Interestingly, Figure 7(b) exhibits the same phenomenon as the “jump clustering” hypothesis proposes. Remark 2 suggests that when ambiguity-averse market makers learn over time, they optimally set the competitive prices at each date, and create clustered jumps to the market prices. In other words, the observed “jump clustering” could be resulted from rational behaviors of ambiguity-averse agents.

Remark 1 shows whether and how ambiguity influence the speed at which information is incorporated into market prices. The following Remark 3 focuses on whether and how the skewness of ambiguity, a key parameter of interest in this paper, affect the speed of information incorporation. Like Figure 6 accompanying Remark 1, all of the following plots report cross-sectional average prices.

**Remark 3** (Ambiguity skewness and the speed of information incorporation) Ambiguity skewness matters to the speed of information incorporation, in the sense that it facilitates the learning from what is more likely to happen but hinders the revelation of the extreme payoffs with small probabilities.

Figure 9 shows that as skewness measure \(q\) increases, the spread and the midpoint converge to zero and the true payoff at a higher speed, respectively. This implies that if the market makers form higher initial belief on the \(GOOD\) world, the information on the payoff associated with the \(GOOD\) world is incorporated into market prices in a faster pace. It is striking to see that in Figure 9(a) even after 1000 trading rounds the spread is still significantly different from zero.

Remark 3 is the dynamic counterpart of Proposition 4. In the static context, Proposition 4 shows that ambiguity-averse market makers tend to set the prices that in turn induce them

\(^{34}\)See, e.g., Benzoni, Collin-Dufresne, Goldstein, and Helwege (2014).
Figure 9: Ambiguity Skewness and Speed of Information Incorporation. We assume 
$q = 0.1$ in figure (a) and $q = 0.9$ in figure (b). We restrict that $qA - (1 - q)B = 0$, and 
that the true payoff is the UP state of the GOOD world. We set $T = 1000$ and simulate 500 
times (cross sections). The bid-ask spread (dashed blue curve), bid-ask midpoint (solid black 
curve), and ask and bid prices (dotted red curves) are all cross-sectional averages. Parameter 
values: $\rho = 0.2$, $r = 0.3$, $\gamma = 3$, $\alpha = 5$, $\sigma = 10$, and $A = 1$. 

Figure (a) Positive Skewness, True State=UP, True World=GOOD 
Figure (b) Negative Skewness, True State=UP, True World=GOOD
to learn from what is more likely to happen. Therefore, taking a longer horizon, Remark 3 shows that the long-term consequence of this pricing rule is that information of the average payoffs with large probabilities gets fast revealed whereas information of the extreme payoffs with small probabilities is hidden.

Remark 3 implies that under the prevalence of ambiguity, the market makers do not necessarily learn from different sources of information at equally slow speed. They learn more and faster from the events with large probabilities but pay less attention to the so-called “tail risk”. This implication is consistent with anecdotal evidence and the literature documenting that crises come from the neglect of the “tail risk” (See, e.g., Gennaioli, Shleifer, and Vishny (2011)).

We further find that the skewness of ambiguity can generate price over-adjustment.

**Remark 4 (Ambiguity skewness and price over-reaction)** Ambiguity skewness may generate price over-reaction to information flows.

Figure 10 shows the dynamic price over-reaction. When \( q = 0.1 \) and the true payoff is in the \((DOWN, GOOD)\) state, the midpoint, as well as the ask and bid prices, first drop then recover. The dynamic price over-reaction is not observed in the case with the same payoff but \( q = 0.5 \). Consistent with Remark 3, after 1000 trading rounds, the midpoint is still different from the true payoff and the spread is still different from zero.

The price over-reaction comes from the rational misinterpretation of information flows. The premise of the misinterpretation is that the states \((DOWN, GOOD)\) and \((DOWN, BAD)\) are close to each other. When sell orders come sequentially at the early dates, the market makers cannot distinguish between the two payoffs. However, they increase the belief on the \(BAD\) world more than the belief on the \(GOOD\) world, because initially at date 0 their belief on the \(BAD\) world is larger. After some threshold (e.g., the bid price is equal to or lower than the payoff in the \((DOWN, GOOD)\) state), the informed traders stop submitting orders, while liquidity traders continue to submit sell order with a non-zero probability. The market makers continue to update the belief on the \(BAD\) world more radically, which can induce price over-reaction. As they receive more zero orders, they realize that the true payoff is more likely to in the \((DOWN, GOOD)\) state, thereby adjusting the midpoint upward gradually. We thus observe that the price first over-react and then recover in the opposite direction.
Figure 10: **Ambiguity Skewness and Price Over-Adjustment.** We assume $q = 0.1$ in figure (a), and $q = 0.5$ in figure (b). We restrict that $qA - (1 - q)B = 0$, and that the true payoff is the DOWN state of the GOOD world. In both panels, we set $T = 1000$ and simulate 500 times (cross sections). The bid-ask spread (dashed blue curve), bid-ask midpoint (solid black curve), and ask and bid prices (dotted red curves) are all cross-sectional averages. Parameter values: $\rho = 0.2$, $r = 0.3$, $\gamma = 3$, $\alpha = 5$, $\sigma = 10$, and $A = 1$. 
6 Comparison with the Maximin Utility

As mentioned in the introduction, the maximin utility is just a special case of the smooth ambiguity utility employed in this paper, therefore the analysis with the maximin utility will miss a lot of implications generated in our setting. For the interest of space, we don’t derive the results under the maximin utility, but only briefly discuss their properties and the reasons why they differ from what we have obtained under the smooth ambiguity utility.

The maximin utility generates multiple equilibria in the static analysis. We assume that the market makers consider two ambiguous worlds, and the smooth ambiguity utility requires them to take both worlds into account. As a result, the market reaches a unique equilibrium (with one bid-ask spread and one bid-ask midpoint) at each date. In contrast, if the market makers have the maximin utility, they take only the worst case into account. However, because in both GOOD and BAD worlds they make zero expected profit, both worlds are the worst cases. For this reason, the market may reach two equilibria, one corresponding to the GOOD world and the other to the BAD world. The indeterminacy of equilibrium under the maximin utility makes our comparative statics impossible.

The maximin utility cannot differentiate ambiguity and ambiguity aversion. A great advantage of the smooth ambiguity utility over the maxmin utility is that the former enables an explicit separation between ambiguity and ambiguity aversion, while they are indistinguishable in the latter. Propositions 1 and 4 show that they have the opposite effects on the equilibrium spread and midpoint.

The maximin utility cannot capture the impact of ambiguity skewness. The smooth ambiguity utility pays attention to both the ambiguous cases and the ambiguous priors of them, thereby enabling the discussion on ambiguity skewness. However, maximin utility focuses only on the ambiguous cases but ignores their subjective probabilities, and as a result, gives no room to the pricing of ambiguity skewness.

The maximin utility produces less tractable dynamic learning. The market makers with the smooth ambiguity utility consider both worlds and all states, and learn from the order flows correctly over time. As a result, the midpoint converges to the true payoff and the spread converges to zero. However, the fact that market makers with the maximin utility consider only the worst case may produce incorrect learning. For example, suppose $A < \sigma$ and the true payoff is $(UP, GOOD)$. If the market makers think the worst case is the GOOD world, then the midpoint converges to the true payoff and the spread converges to zero over time. Nevertheless, if they treat the BAD world as the worst case, the midpoint converges to the wrong payoff (i.e. $(UP, BAD)$) no matter how long they learn. In another example, suppose again $A < \sigma$, the true payoff is $(DOWN, GOOD)$ and the market makers treat
the BAD world as the worst case. No matter how long they learn, the midpoint wanders between \((UP, BAD)\) and \((DOWN, BAD)\), and the spread does not converge to zero. In the latter example, the market makers hold the incorrect belief, but they have no way to correct it through dynamic learning.

7 Concluding Remarks

Inspired by the observation that ambiguity plays a significant role in affecting market liquidity, we develop a multi-period trading model à la Glosten and Milgrom (1985), where ambiguity-averse market makers, informed traders, and price-insensitive liquidity traders trade sequentially and the market makers update their priors through dynamic learning. A measure of ambiguity skewness is introduced to indicate the relative salience of upside payoffs to downside payoffs. Our parsimonious and tractable model generates novel findings that relate ambiguity, ambiguity aversion, and ambiguity skewness to market liquidity, return predictability, and price efficiency.

For market liquidity, we find that in general, higher ambiguity leads to higher illiquidity, but there exist some thresholds at which market liquidity suddenly dries up or floods. Based on these findings, we propose the linkage between ambiguity and the sudden changes in market liquidity. Interestingly, we can use the model’s predictions to interpret what we have witnessed during the shorting ban event, the market freezes of 2007-2009 financial crisis, and the “Flash Crash” of May 6, 2010.

For return predictability, we find that ambiguity skewness on an asset’s payoff predicts the sign of its future return. More precisely, positive skewness predicts a negative return, while negative skewness predicts a positive return. We also find that ambiguity skewness determines the magnitude by which ambiguity explains contemporaneous return and predicts future return. For example, when the ambiguity is positively skewed, higher ambiguity produces a higher contemporaneous return and a lower future returns. The ambiguity aversion channel of this return predictability is distinct from the optimal expectations and the probability weighting channels proposed in the literature. Our findings highlight both ambiguity and ambiguity skewness as new factors to determine asset prices and provide new testable hypotheses.

For price efficiency, we find that ambiguity lowers the speed of learning, that moderate ambiguity produces price “jump clustering”, and that ambiguity skewness impedes the revelation of “tail risk” and may produce dynamic price over-reaction. We therefore suggest that the social cost of ambiguity contains not only illiquidity, but also price inefficiency, as well as creation of the “jump risk” and amplification of the “tail risk”.

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We emphasize that one of our contributions is to document and investigate the separation and interaction between ambiguity and risk. We are the first to show that bid-ask spread contains a risk premium and an ambiguity premium, and we find that under symmetric ambiguity ambiguity premium is increasing in ambiguity but decreasing in risk aversion. We demonstrate that ambiguity and ambiguity aversion have opposite impact on asset prices and market liquidity. Importantly, we find that the salience of ambiguity relative to risk is a necessary condition to liquidity dry-ups and flooding. If the distance between zero and the liquidity dry-up ambiguity threshold is used to measure the resistance to ambiguity-induced liquidity dry-up, we can observe that the resistance increases in risk. The policy implication from this discussion is that if our goal is to create the highest market liquidity, we should make the best effort in lowering ambiguity. However, if our resources are limited and our goal is at least to prevent liquidity dry-up, we should allocate more effort to less risky assets and less effort to more risky assets.

In this paper we assume that the informed traders are perfectly informed, but one can prove that our result is robust if their information is noisy. The key requirement behind our findings is that the informed traders have information advantage over the market makers, so that the latter have to set the prices to compensate for adverse selection cost.

We also assume that the liquidity traders are inelastic to the bid and ask prices. If we relax this assumption by allowing them choose not to trade when the bid-ask spread widens, our results will be strengthened. Under the new assumption of price-sensitive liquidity traders, we can solve a liquidity freeze threshold. When ambiguity is smaller than the threshold, market liquidity deteriorates as the ambiguity increases. When the ambiguity is equal to or larger than the threshold, liquidity traders exit the market and the only counterparties of the market makers are the informed traders. As a consequence, the market makers will set the bid and ask prices to make the informed traders feel indifferent between trading and exiting. Therefore the market freezes if the ambiguity is larger than the liquidity freeze threshold.

Because our findings help to reconcile some conflicting empirical results, we are keen to look for a refinement of existing proxies for ambiguity and ambiguity skewness and to confront the model’s implications with return data directly in future studies.
References


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Appendix: Equilibrium Analysis in the Static Setting

Proof of Proposition 1. Given the analysis in section 3.1, we can solve $M_1^*$ and $S_1^*$ by the following two equations

\[ q(e^\gamma \rho (A-M_1^*) + e^{-\gamma \rho (A-M_1^*)}) + (1-q)(e^\gamma \rho (B+M_1^*) + e^{-\gamma \rho (B+M_1^*)}) = 0, \quad (A1) \]

and

\[ e^{\alpha \gamma [\rho \sigma - (\rho+2r)S_1^*]}[q(e^\gamma \rho (A-M_1^*) + e^{-\gamma \rho (A-M_1^*)}) + (1-q)(e^\gamma \rho (B+M_1^*) + e^{-\gamma \rho (B+M_1^*)})^\alpha] = 2^\alpha. \quad (A2) \]

Let $X_1^+ = e^\gamma \rho (A-M_1^*) + e^{-\gamma \rho (A-M_1^*)} > 0$, $X_2^+ = e^\gamma \rho (B+M_1^*) + e^{-\gamma \rho (B+M_1^*)} > 0$, $X_1^- = e^\gamma \rho (A-M_1^*) - e^{-\gamma \rho (A-M_1^*)} < 0$, and $X_2^- = e^\gamma \rho (B+M_1^*) - e^{-\gamma \rho (B+M_1^*)} > 0$. Taking first-order derivative of equation (A1) on both sides w.r.t $A$ gives

\[ -q(1 - \frac{\partial M_1^*}{\partial A})[(X_1^+)^\alpha + (\alpha - 1)(X_1^-)^2(X_1^+)^{\alpha-2}] + (1-q)(\frac{q}{1-q} + \frac{\partial M_1^*}{\partial A})[(X_2^+)^\alpha + (\alpha - 1)(X_2^-)^2(X_2^+)^{\alpha-2}] = 0, \quad (A3) \]

which implies $\frac{\partial M_1^*}{\partial A} \in (-\frac{q}{1-q}, 1)$. Taking first-order derivative of equation (A2) on both sides w.r.t $A$ gives

\[ \alpha \gamma (\rho + 2r)e^{\alpha \gamma [\rho \sigma - (\rho+2r)S_1^*]}[q(X_1^+)^\alpha + (1-q)(X_2^+)^\alpha]\frac{\partial S_1^*}{\partial A} = \alpha \gamma pe^{\alpha \gamma [\rho \sigma - (\rho+2r)S_1^*]}[q(1 - \frac{\partial M_1^*}{\partial A})(-X_1^-)(X_1^+)^{\alpha-1} + (1-q)(\frac{q}{1-q} + \frac{\partial M_1^*}{\partial A})](X_2^-)(X_2^+)^{\alpha-1} > 0, \]

which proves

\[ \frac{\partial S_1^*}{\partial A} > 0. \]

Taking first-order derivative of equation (A2) on both sides w.r.t $\alpha$ gives

\[ e^{\alpha \gamma [\rho \sigma - (\rho+2r)S_1^*]}[q(X_1^+)^\alpha + (1-q)(X_2^+)^\alpha][\gamma(\rho \sigma - (\rho+2r)S_1^*) - \alpha \gamma (\rho + 2r)\frac{\partial S_1^*}{\partial \alpha}] + e^{\alpha \gamma [\rho \sigma - (\rho+2r)S_1^*]}[q(X_1^+)^\alpha \ln(X_1^+) + q\alpha \gamma \rho \frac{\partial M_1^*}{\partial \alpha}(X_1^-)(X_1^+)^{\alpha-1} + (1-q)(X_2^+)^\alpha \ln(X_2^+) + (1-q)\alpha \gamma \rho \frac{\partial M_1^*}{\partial \alpha}(X_2^-)(X_2^+)^{\alpha-1}] = 2^\alpha \ln 2. \]

Because of equation (A1), the above equation is equivalent to

\[ e^{\alpha \gamma [\rho \sigma - (\rho+2r)S_1^*]}[q(X_1^+)^\alpha + (1-q)(X_2^+)^\alpha][\gamma(\rho \sigma - (\rho+2r)S_1^*) - \alpha \gamma (\rho + 2r)\frac{\partial S_1^*}{\partial \alpha}] + e^{\alpha \gamma [\rho \sigma - (\rho+2r)S_1^*]}[q(X_1^+)^\alpha \ln(X_1^+) + (1-q)(X_2^+)^\alpha \ln(X_2^+)] = 2^\alpha \ln 2, \]

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or

\[ 2^\alpha[\gamma(\rho \sigma - (\rho + 2r)S_1^*) - \alpha \gamma(\rho + 2r)\frac{\partial S_1^*}{\partial \alpha}] + \Theta(q) = 2^\alpha \ln 2, \]

where \( \Theta(q) = e^{\alpha \gamma(\rho \sigma - (\rho + 2r)S_1^*)\cancel{q}(X_1^+ \ln(X_1^*) + (1-q)(X_2^+)^\alpha \ln(X_2^+))}. \) Taking first-order derivative of the previous equation on both sides w.r.t. \( q \) yields

\[ \gamma(\rho + 2r)2^\alpha \frac{\partial S_1^*}{\partial q} + \alpha \gamma(\rho + 2r)2^\alpha \frac{\partial^2 S_1^*}{\partial \alpha \partial q} = \frac{\partial \Theta(q)}{\partial q}. \quad (A4) \]

From equation (A2) we can easily derive that \( \frac{\partial S_1^*}{\partial q} |_{q=\frac{1}{2}} = 0, \frac{\partial S_1^*}{\partial q} |_{q>\frac{1}{2}} < 0, \) and \( \frac{\partial S_1^*}{\partial q} |_{q<\frac{1}{2}} > 0. \)

Suppose \( q = \frac{1}{2} + \Delta, \) where \( \Delta > 0 \) is extremely small. Because we have \( X_2^+(q = \frac{1}{2} + \Delta) > X_1^+(q = \frac{1}{2}) > X_1^+(q = \frac{1}{2}), \) we can easily derive that \( \frac{\partial \Theta(q)}{\partial q} |_{q=\frac{1}{2}} = 0, \frac{\partial \Theta(q)}{\partial q} |_{q>\frac{1}{2}} > 0, \) and \( \frac{\partial \Theta(q)}{\partial q} |_{q<\frac{1}{2}} < 0. \) Therefore, combining the properties of \( \frac{\partial S_1^*}{\partial q} \) and \( \frac{\partial \Theta(q)}{\partial q} \) with equation (A4) gives

\[
\begin{align*}
\text{When } q &= \frac{1}{2}, \quad \frac{\partial^2 S_1^*}{\partial \alpha \partial q} = 0; \\
\text{When } q &> \frac{1}{2}, \quad \frac{\partial^2 S_1^*}{\partial \alpha \partial q} > 0; \\
\text{When } q &< \frac{1}{2}, \quad \frac{\partial^2 S_1^*}{\partial \alpha \partial q} < 0.
\end{align*}
\]

Because we have shown that \( \frac{\partial S_1^*}{\partial \alpha} |_{q=\frac{1}{2}} = 0, \) we have \( \frac{\partial S_1^*}{\partial \alpha} \geq 0, \) or

\[ \frac{\partial S_1^*}{\partial (1-\alpha)} \leq 0, \]

where the equality holds only when \( q = \frac{1}{2}. \) \( \blacksquare \)

**Proof of Proposition 2.** To characterize the full equilibria, we need to consider 2 cases and 4 subcases:

**Case (a):** \( A \leq \sigma \) (so that \( -A + \sigma \geq A - \sigma \)).

**Subcase (a-1):** if it turns out to be \( b_1^* = M_1^* - S_1^* \geq A - \sigma \) and \( a_1^* = M_1^* + S_1^* \leq -A + \sigma, \) Therefore,

\[
\begin{align*}
U = &E[\phi(E[u(w)])] \\
= &- \frac{1}{\alpha} \frac{1}{2} 2^\alpha e^{\alpha \gamma(\rho \sigma - (\rho + 2r)S_1)} (e^{\gamma \rho(A-M_1)} + e^{-\gamma \rho(A-M_1)})^\alpha \\
&- \frac{1}{\alpha} \frac{1}{2} 2^\alpha e^{\alpha \gamma(\rho \sigma - (\rho + 2r)S_1)} (e^{\gamma \rho(A+M_1)} + e^{-\gamma \rho(A+M_1)})^\alpha \\
= &- \frac{1}{\alpha} 2^\alpha e^{\alpha \gamma(\rho \sigma - (\rho + 2r)S_1)} \left[ \frac{1}{2} (e^{\gamma \rho(A-M_1)} + e^{-\gamma \rho(A-M_1)})^\alpha \\
&+ \frac{1}{2} (e^{\gamma \rho(A+M_1)} + e^{-\gamma \rho(A+M_1)})^\alpha \right] \\
= &- \frac{1}{\alpha} \quad (A5)
\end{align*}
\]
Define \( g_1(M_1) = \frac{1}{2}(e^{\gamma(A-M_1)} + e^{-\gamma(A-M_1)})^\alpha + \frac{1}{2}(e^{\gamma(A+M_1)} + e^{-\gamma(A+M_1)})^\alpha \). Then we have

\[
\frac{\partial g_1}{\partial M_1} = \frac{1}{2}\alpha\gamma\rho(e^{\gamma(A-M_1)} + e^{-\gamma(A-M_1)})\alpha - 1(e^{-\gamma(A-M_1)} - e^{\gamma(A-M_1)}) + \frac{1}{2}\alpha\gamma\rho(e^{\gamma(A+M_1)} + e^{-\gamma(A+M_1)})\alpha - 1(e^{\gamma(A+M_1)} - e^{-\gamma(A+M_1)}),
\]

and

\[
\frac{\partial^2 g_1}{\partial M_1^2} = \frac{1}{2}\alpha\gamma \rho^2(e^{\gamma(A-M_1)} + e^{-\gamma(A-M_1)})\alpha - 2(e^{-\gamma(A-M_1)} - e^{\gamma(A-M_1)})^2 + \frac{1}{2}\alpha\gamma \rho^2(e^{\gamma(A+M_1)} + e^{-\gamma(A+M_1)})\alpha - 2(e^{\gamma(A+M_1)} - e^{-\gamma(A+M_1)})^2 + \frac{1}{2}\alpha\gamma \rho^2(e^{\gamma(A+M_1)} + e^{-\gamma(A+M_1)})\alpha.
\]

Because \( \frac{\partial g_1}{\partial M_1} |_{M_1=0} = 0 \), \( \frac{\partial^2 g_1}{\partial M_1^2} > 0 \), when \( M_1 = 0 \), \( S_1 \) reaches its minimum. From equation (A5), we have

\[
e^{\gamma[(\rho - (\rho+2r))S_1^*] + e^{-\gamma(A)} + e^{-\gamma(A)}} = 2. \tag{A6}
\]

Therefore, we obtain

\[
S_1^* = \frac{\rho}{\rho + 2r} - \sigma + \frac{\ln(e^{\gamma(A)} + e^{-\gamma(A)}) - \ln 2}{(\rho + 2r)\gamma}, \tag{A7}
\]

\[
M_1^* = 0.
\]

Because \( b_1^* = M_1^* - S_1^* \geq A - \sigma \), or

\[
0 - \frac{\rho}{\rho + 2r} - \sigma - \ln(e^{\gamma(A)} + e^{-\gamma(A)}) - 2\ln 2 \geq A - \sigma,
\]

which gives

\[
(\rho + 2r)A + \frac{\ln(e^{\gamma(A)} + e^{-\gamma(A)}) - \ln 2}{\gamma} \leq 2r\sigma.
\]

So \( A \in (0, A_d) \), where

\[
(\rho + 2r)A_d + \frac{\ln(e^{\gamma(A_d)} + e^{-\gamma(A_d)}) - \ln 2}{\gamma} = 2r\sigma,
\]

and \( A_d < \sigma \).

The leftmost spread curve in Figure 4 is determined by equation (A7).

**Subcase (a-2):** If it turns out to be \( b_1^* = M_1^* - S_1^* < A - \sigma \) and \( a_1^* = M_1^* + S_1^* > -A + \sigma \), then the expected utility conditional on the GOOD world is

\[
E[u(w)|GOOD world| = -\frac{1}{2}\exp\{-\gamma[(\rho + r)(M_1 + S_1 - A - \sigma) + (r)(A + \sigma - M_1 + S_1)]\} - \frac{1}{2}\exp\{-\gamma[(r)(A - \sigma - M_1 + S_1) + (r)(M_1 + S_1 - A + \sigma)]\}
\]

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Therefore
\[\frac{\partial g_2}{\partial M_1} = \frac{1}{\alpha} \frac{\partial}{\partial M_1} \left[ \frac{e^{-\gamma[(\rho+2r)S_1 + \rho(M_1 - A) - \rho\sigma]]} + e^{-\gamma \cdot 2r S_1}}{\alpha} \right] - 1 \]

And
\[\frac{\partial g_2}{\partial S_1} = - \left\{ g[(\rho + 2r)] e^{-\gamma[(\rho+2r)S_1 + \rho(M_1 - A) - \rho\sigma] + (\gamma \cdot 2r)e^{-\gamma \cdot 2r S_1}] \right\} \]
\[\cdot \left[ e^{-\gamma[(\rho+2r)S_1 + \rho(M_1 - A) - \rho\sigma]} + e^{-\gamma \cdot 2r S_1}] \alpha^{-1} \right] - 1 \]
\[\cdot \left( \left\{ \gamma \cdot 2r \right\} e^{-\gamma \cdot 2r S_1} + \left\{ \gamma \cdot 2r \right\} e^{-\gamma[(\rho+2r)S_1 + \rho(M_1 + A) - \rho\sigma]} \right) \alpha^{-1} \]

Therefore, we have
\[e^{-\gamma \cdot 2r S_1} + e^{-\gamma[(\rho+2r)S_1 + \rho(M_1 + A) - \rho\sigma]} = 2. \]
It can be easily proved that
\[ S_1^* > \frac{\rho}{\rho + 2r} \sigma + \frac{\ln(e^{\gamma_A} + e^{-\gamma_A}) - \ln 2}{(\rho + 2r) \gamma}, \]  
(A10)

where \( A \in (A_d, \sigma) \).

Combining equation (A7) and equation (A10) shows the only Nash equilibrium is that when \( A \in (0, A_d] \), \( S_1^* = \frac{\rho}{\rho + 2r} \sigma + \frac{\ln(e^{\gamma_A} + e^{-\gamma_A}) - \ln 2}{(\rho + 2r) \gamma} \), and when \( A \in (A_d, \sigma] \), \( S_1^* > \frac{\rho}{\rho + 2r} \sigma + \frac{\ln(e^{\gamma_A} + e^{-\gamma_A}) - \ln 2}{(\rho + 2r) \gamma} \).

In other words, there is a discontinuous jump of \( S_1^* \) at the point \( A = A_d \in (0, \sigma) \). We call \( A_d \) the liquidity dry-up threshold.

The middle spread curve in Figure 4 is determined by equation (A9).

**Case (b):** \( A > \sigma \) (so \( -A + \sigma < A - \sigma \)).

**Subcase (b-1):** if it turns out to be \( b_1^* = M_1^* - S_1^* \geq -A + \sigma \) and \( a_1^* = M_1^* + S_1^* \leq A - \sigma \), then the expected utility conditional on the GOOD world is
\[ E[u(w)|\text{GOOD world}] = -\frac{1}{2} \exp\{-\gamma[(\rho + r)(M_1 + S_1 - A - \sigma) + (r)(A + \sigma - M_1 + S_1)]\} \]
\[ -\frac{1}{2} \exp\{-\gamma[(\rho + r)(M_1 + S_1 - A + \sigma) + (r)(A - \sigma - M_1 + S_1)]\} \]
\[ = -\frac{1}{2} e^{-\gamma[(\rho + 2r)S_1 + \rho(M_1 - A)]}[e^{\gamma_A} + e^{-\gamma_A}]. \]

and the expected utility conditional on the BAD world is
\[ E[u(w)|\text{BAD world}] = -\frac{1}{2} \exp\{-\gamma[(\rho + r)(-A + \sigma - M_1 + S_1) + (r)(M_1 + S_1 + A - \sigma)]\} \]
\[ -\frac{1}{2} \exp\{-\gamma[(\rho + r)(-A - \sigma - M_1 + S_1) + (r)(M_1 + S_1 + A + \sigma)]\} \]
\[ = -\frac{1}{2} e^{-\gamma[(\rho + 2r)S_1 - \rho(M_1 + A)]}[e^{\gamma_A} + e^{-\gamma_A}]. \]

Therefore,
\[ U = E[\phi(E[u(w)])] \]
\[ = -\frac{1}{2} \frac{1}{\alpha} \frac{1}{2^\alpha} e^{-\alpha \gamma[(\rho + 2r)S_1 + \rho(M_1 - A)]}[e^{\gamma_A} + e^{-\gamma_A}] \]
\[ -\frac{1}{2} \frac{1}{\alpha} \frac{1}{2^\alpha} e^{-\alpha \gamma[(\rho + 2r)S_1 - \rho(M_1 + A)]}[e^{\gamma_A} + e^{-\gamma_A}] \]
\[ = -\frac{1}{2} \frac{1}{\alpha} \frac{1}{2^\alpha} e^{-\alpha \gamma[(\rho + 2r)S_1 - \rho A]}[e^{\gamma_A} + e^{-\gamma_A}] [\frac{1}{2} e^{-\alpha \gamma A} + \frac{1}{2} e^{\alpha \gamma A}] \]
\[ = -\frac{1}{\alpha}. \]  
(A11)

It can be easily proven that when \( M_1 = 0, S_1 \) reaches its minimum. Then from equation (A11) we have
\[ e^{-\gamma[(\rho + 2r)S_1 - \rho A](e^{\gamma_A} + e^{-\gamma_A})} = 2. \]  
(A12)
Therefore, we solve

\[ S_1^* = \frac{\rho}{\rho + 2r} A + \frac{\ln(e^{\gamma \rho a} + e^{-\gamma \rho a}) - \ln 2}{(\rho + 2r) \gamma}, \quad (A13) \]

\[ M_1^* = 0. \]

Because \( b_1^* = M_1^* - S_1^* \geq -A + \sigma \), or

\[ 0 - \frac{\rho}{\rho + 2r} A - \frac{\ln(e^{\gamma \rho a} + e^{-\gamma \rho a}) - \ln 2}{(\rho + 2r) \gamma} \geq -A + \sigma, \]

which gives

\[ 2rA \geq (\rho + 2r)\sigma + \frac{\ln(e^{\gamma \rho a} + e^{-\gamma \rho a}) - \ln 2}{\gamma}. \]

So \( A \in [A_f, +\infty) \), where

\[ 2rA_f = (\rho + 2r)\sigma + \frac{\ln(e^{\gamma \rho a} + e^{-\gamma \rho a}) - \ln 2}{\gamma}, \]

and \( A_f > \sigma \).

The rightmost spread curve in Figure 4 is determined by equation (A13).

**Subcase (b-2):** if it turns out to be \( b_1^* = M_1^* - S_1^* < -A + \sigma \) and \( a_1^* = M_1^* + S_1^* > A - \sigma \), then the expected utility conditional on the *GOOD* world is

\[
E[u(w)|\text{GOOD world}] = -\frac{1}{2} \exp\{-\gamma[(\rho + r)(M_1 + S_1 - A - \sigma) + (r)(A + \sigma - M_1 + S_1)]\} - \frac{1}{2} \exp\{-\gamma[(r)(A - \sigma - M_1 + S_1) + (r)(M_1 + S_1 - A + \sigma)]\} = -\frac{1}{2} \exp\{-\gamma[(\rho + 2r)S_1 + \rho(M_1 - A) - \rho\sigma]\} - \frac{1}{2} \exp\{-\gamma \cdot 2rS_1\},
\]

and the expected utility conditional on the *BAD* world is

\[
E[u(w)|\text{BAD world}] = -\frac{1}{2} \exp\{-\gamma[(\rho + r)(M_1 + S_1 + A - \sigma) + (r)(-A + \sigma - M_1 + S_1)]\} - \frac{1}{2} \exp\{-\gamma[(\rho + r)(-A - \sigma - M_1 + S_1) + (r)(M_1 + S_1 + A + \sigma)]\} = -\frac{1}{2} \exp\{-\gamma \cdot 2rS_1\} - \frac{1}{2} \exp\{-\gamma[(\rho + 2r)S_1 - \rho(M_1 + A) - \rho\sigma]\}.
\]

Therefore

\[
U = E[\phi(E[u(w)])] = -\frac{1}{2} \frac{1}{2^\alpha \alpha^{2\alpha}} \left[e^{-\gamma[(\rho + 2r)S_1 + \rho(M_1 - A) - \rho\sigma]} + e^{-\gamma 2rS_1}\right] - \frac{1}{2} \frac{1}{2^\alpha \alpha^{2\alpha}} \left[e^{-\gamma 2rS_1} + e^{-\gamma[(\rho + 2r)S_1 - \rho(M_1 + A) - \rho\sigma]}\right] = -\frac{1}{\alpha}, \quad (A14)
\]
Therefore
\[
\frac{\partial g_4}{\partial M_1} = - (r \rho) e^{-\gamma[(\rho + 2r)S_1 + \rho(M_1 - A) - \rho \sigma]} e^{-\gamma 2rS_1} + (\rho + 2r) e^{-\gamma 2rS_1} \alpha - 1
\]
and
\[
\frac{\partial g_4}{\partial S_1} = - \{ [\gamma (\rho + 2r)] e^{-\gamma[(\rho + 2r)S_1 + \rho(M_1 - A) - \rho \sigma]} + (\gamma \cdot 2r) e^{-\gamma 2rS_1} \}
\cdot \left[ e^{-\gamma[(\rho + 2r)S_1 + \rho(M_1 - A) - \rho \sigma]} + e^{-\gamma 2rS_1} \right] \alpha - 1
\]
\[
- \{ [\gamma \cdot 2r] e^{-\gamma 2rS_1} + [\gamma (\rho + 2r)] e^{-\gamma[(\rho + 2r)S_1 + \rho(M_1 + A) - \rho \sigma]} \}
\cdot \left[ e^{-\gamma 2rS_1} + e^{-\gamma[(\rho + 2r)S_1 + \rho(M_1 + A) - \rho \sigma]} \right] \alpha - 1
\]
\[
< 0.
\]
Let \( \frac{\partial S_1}{\partial M_1} = - \frac{\partial g_4}{\partial S_1} = 0 \), so we have \( \frac{\partial g_4}{\partial M_1} = 0 \), which gives \( M_1^* = 0 \).

From equation (A14) we have,
\[
e^{-\gamma 2rS_1^*} + e^{-\gamma[(\rho + 2r)S_1^* - \rho A - \rho \sigma]} = 2. \tag{A15}
\]

It can be easily proved that
\[
S_1^* > \frac{\rho}{\rho + 2r} A + \frac{\ln(e^{\gamma \rho \sigma} + e^{-\gamma \rho \sigma}) - \ln 2}{(\rho + 2r) \gamma}, \tag{A16}
\]
where \( A \in (\sigma, A_f) \).

Combining equation (A13) and equation (A16) shows the only Nash equilibrium is that when \( A \in (\sigma, A_f) \), \( S_1^* > \frac{\rho}{\rho + 2r} A + \frac{\ln(e^{\gamma \rho \sigma} + e^{-\gamma \rho \sigma}) - \ln 2}{(\rho + 2r) \gamma} \), and when \( A \in [A_f, +\infty) \), \( S_1^* = \frac{\rho}{\rho + 2r} A + \frac{\ln(e^{\gamma \rho \sigma} + e^{-\gamma \rho \sigma}) - \ln 2}{(\rho + 2r) \gamma} \).

In other words, there is a discontinuous drop of \( S_1^* \) at the point \( A = A_f \in (\sigma, +\infty) \). We call \( A_f \) the liquidity flooding threshold.

**Proof of Proposition 3.** From equation (3) we can derive
\[
\frac{1}{\gamma \rho} \frac{\partial^2 f}{\partial M_1 \partial q} = \alpha (e^{\gamma \rho(A - M_1)} + e^{-\gamma \rho(A - M_1)}) \alpha - 1 (e^{-\gamma \rho(A - M_1)} - e^{\gamma \rho(A - M_1)})
\]
\[
- \frac{\alpha (e^{\gamma \rho(B + M_1)} + e^{-\gamma \rho(B + M_1)}) \alpha - 1 (e^{\gamma \rho(B + M_1)} - e^{-\gamma \rho(B + M_1)})}{1 - \frac{A \gamma \rho}{\alpha - 1} (e^{\gamma \rho(B + M_1)} + e^{-\gamma \rho(B + M_1)}) \alpha - 2 (e^{\gamma \rho(B + M_1)} - e^{-\gamma \rho(B + M_1)})^2}
\]
Let $Q = A\gamma \rho$, then we have

$$
\frac{1}{\gamma \rho} \frac{\partial^2 f}{\partial M_1 \partial q} \bigg|_{M_1=0,q=\frac{1}{2}} = 2\alpha (\alpha - 1)Q(e^Q + e^{-Q})^{\alpha-2}(e^Q - e^{-Q})^2 + 2\alpha Q(e^Q + e^{-Q})^\alpha - 2\alpha(e^Q + e^{-Q})^{\alpha-1}(e^Q - e^{-Q})
$$

where $z(Q) = (\alpha - 1)(e^Q - e^{-Q})^2 Q + (e^Q + e^{-Q})^2 Q - (e^Q + e^{-Q})(e^Q - e^{-Q})$. Because $z(0) = 0$, then it can be shown that for any $\alpha > 1$ and $Q > 0$,

$$
\frac{\partial z(Q)}{Q} = 2\alpha Q(e^Q + e^{-Q})(e^Q - e^{-Q}) + (\alpha - 2)(e^Q - e^{-Q})^2 > 0.
$$

Therefore we have $z(Q) > 0$ for $Q > 0$. Because $A\gamma \rho > 0$, $\frac{\partial^2 f}{\partial M_1 \partial q} \bigg|_{M_1=0,q=\frac{1}{2}} > 0$. And because

$$
\frac{\partial f}{\partial M_1} \bigg|_{M_1=0,q=\frac{1}{2}} = 0, \frac{\partial^2 f}{\partial M_1^2} > 0. \text{ So we have}
$$

$$
\frac{\partial M_1^*}{\partial q} \bigg|_{M_1=0,q=\frac{1}{2}} = - \frac{\partial^2 f}{\partial M_1 \partial q} \bigg|_{M_1=0,q=\frac{1}{2}} < 0,
$$

which implies that (1) when $q = \frac{1}{2}$, $M_1^* = \frac{A-B}{2} = 0$; (2) when $q > \frac{1}{2}$, $M_1^* < 0$; (3) when $q < \frac{1}{2}$, $0 < M_1^*$.

From equation (3), we have

$$
\frac{1}{\gamma \rho} \frac{\partial f}{\partial M_1} \bigg|_{M_1=\frac{A-B}{2}} = (1 - 2q)\alpha(e^{\gamma \rho \frac{A-B}{2}} + e^{-\gamma \rho \frac{A-B}{2}})^{\alpha-1}(e^{\gamma \rho \frac{A-B}{2}} - e^{-\gamma \rho \frac{A-B}{2}}).
$$

Because we know $(e^{\gamma \rho \frac{A-B}{2}} + e^{-\gamma \rho \frac{A-B}{2}})^{\alpha-1}(e^{\gamma \rho \frac{A-B}{2}} - e^{-\gamma \rho \frac{A-B}{2}}) > 0$, we can conclude that when $q < \frac{1}{2}$, $\frac{\partial f}{\partial M_1} \bigg|_{M=\frac{A-B}{2}} > 0$ implies $M_1^* < \frac{A-B}{2}$, and that when $q > \frac{1}{2}$, $\frac{1}{\gamma \rho} \frac{\partial f}{\partial M_1} \bigg|_{M=\frac{A-B}{2}} < 0$ implies $M_1^* > \frac{A-B}{2}$.

To sum up, we have (1) when $q = \frac{1}{2}$, $M_1^* = \frac{A-B}{2} = 0$; (2) when $q > \frac{1}{2}$, $\frac{A-B}{2} < M_1^* < 0$; (3) when $q < \frac{1}{2}$, $0 < M_1^* < \frac{A-B}{2}$. ■

**Proof of Proposition 4.** Taking first-order derivative of equation (A1) on both sides w.r.t $\alpha$ gives

$$
\gamma \rho \frac{\partial M_1^*}{\partial \alpha} \left[q(X_1^+)^\alpha + (\alpha - 1)(X_1^-)^2(X_2^+)^{\alpha-2} + (1 - q)(X_2^+)^\alpha + (1 - q)(\alpha - 1)(X_2^-)^2(X_2^+)^{\alpha-2}\right]
$$

$$
+ q(X_1^-)(X_2^+)^{\alpha-1} \ln(X_1^+) + (1 - q)(X_2^-)(X_2^+)^{\alpha-1} \ln(X_2^+) = 0,
$$

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We solve the first liquidity dry-up threshold when we can solve

Then we solve the four possible thresholds in the following five subcases when the bid-ask range

or

where \( \Psi_1 = \gamma \rho [g(X_1^+)^\alpha + q(\alpha - 1)X_1^-] + (1 - q)(\alpha - 1)(X_1^-)^2 + (1 - q)(\alpha - 1)(X_2^+)^2 + (1 - q)(\alpha - 1)(X_2^-)^2 > 0, \)

and \( \Psi_2 = q(X_1^-)(X_1^+)^\alpha - 1 \ln(X_1^-) + (1 - q)(X_2^-)(X_2^+)^\alpha - 1 \ln(X_2^-) \). It can be easily proved that when

\( q = \frac{1}{2}, \Psi_2 = 0, \) when \( q < \frac{1}{2}, \Psi_2 < 0, \) and when \( q > \frac{1}{2}, \Psi_2 > 0. \) Therefore we conclude that (1) when \( q = \frac{1}{2}, \frac{\partial M_1^*}{\partial (1 - \alpha)} = 0, \) (2) when \( q < \frac{1}{2}, \frac{\partial M_1^*}{\partial (1 - \alpha)} < 0, \) (1) when \( q > \frac{1}{2}, \frac{\partial M_1^*}{\partial (1 - \alpha)} > 0. \)

From equation (A3), it is straightforward to prove that (1) when \( q = \frac{1}{2}, \frac{\partial M_1^*}{\partial M} = 0, \) (2) when \( q < \frac{1}{2}, \frac{\partial M_1^*}{\partial M} > 0, \) (1) when \( q > \frac{1}{2}, \frac{\partial M_1^*}{\partial M} < 0. \)

**Proof of Proposition 5.** We first guess that there are four possible thresholds for liquidity jumps. Then we solve the four possible thresholds in the following five subcases when the bid-ask range falling in different intervals. We show the proof with \( q > \frac{1}{2}, \) and the proof with \( q < \frac{1}{2} \) is similar and skipped. For all of the proof below, we eliminate the subscripts 1 and the superscripts \( * \) of \( b_1, \ a_1, \ M_1, \) and \( S_1 \) for simplicity. Instead we use \( b, \ a, \ M, \) and \( S \) to represent the equilibrium bid price, ask price, bid-ask midpoint, and bid-ask spread in case \((i)\), where \( i = a, \ b, \ c, \ d, \) and \( e, \) as illustrated in Figure 11.

To give the key equations in the five cases.

**Case (a):** if it turns out to be \( b_a = M_a - S_a \geq A - \sigma \) and \( a_a = M_a + S_a \leq -B + \sigma, \) then we can solve \( M_a \) and \( S_a \) by equations (A19) and (A20):

\[
\begin{align*}
(q)[e^{-\gamma \rho (A - M_a)} + e^{-\gamma \rho (A - M_a)}][e^\gamma \rho (A - M_a) + e^{-\gamma \rho (A - M_a)}]^\alpha - 1 \\
+ (1 - q)[e^{-\gamma \rho (B + M_a)} + e^{-\gamma \rho (B + M_a)}][e^{-\gamma \rho (B + M_a)} + e^{-\gamma \rho (B + M_a)}]^\alpha - 1 = 0, \tag{A19}
\end{align*}
\]

and

\[
\begin{align*}
e^{\alpha \gamma [\rho - (\rho + 2r)S_a]} \{ (q)[e^{-\gamma \rho (A - M_a)} + e^{-\gamma \rho (A - M_a)}]^\alpha \\
+ (1 - q)[e^{-\gamma \rho (B + M_a)} + e^{-\gamma \rho (B + M_a)}]^\alpha \} = 2^\alpha. \tag{A20}
\end{align*}
\]

We solve the first liquidity dry-up threshold \( A_{d1} \) by the following equation (A21) given \( q: \)

\[
M_a(q, A_{d1}) + S_a(q, A_{d1}) = -B + \sigma. \tag{A21}
\]

Equations (A19) and (A20) determine the leftmost segment of bid-ask midpoint and spread curves, respectively.

**Case (b):** if it turns out to be \( b_b = M_b - S_b \geq A - \sigma \) and \( a_b = M_b + S_b \geq -B + \sigma, \) then we can solve \( M_b \) and \( S_b \) by equations (A22) and (A23):

\[
\begin{align*}
(q)[e^{\alpha \gamma [\rho - (\rho + 2r)S_b]}[e^{-\gamma \rho (A - M_b)} - e^{-\gamma \rho (A - M_b)}][e^{-\gamma \rho (A - M_b)} + e^{-\gamma \rho (A - M_b)}]^\alpha - 1 \\
- (1 - q)e^{-\gamma [(\rho + 2r)S_b - \rho (M_b + B) - \rho \sigma]}[e^{-\gamma (2rS_b) + e^{-\gamma [(\rho + 2r)S_b - \rho (M_b + B) - \rho \sigma)]}]^\alpha - 1 = 0, \tag{A22}
\end{align*}
\]

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Figure 11: Increasing Asymmetric Ambiguity. Without loss of generality, assume \( q > 1/2 \). As ambiguity increases from zero to infinity, the pricing rule evolves from figure (a) to figure (e).
We solve the first liquidity flooding threshold \( A_{f1} \) by the following equation (A24) given \( q \):

\[
M^b(q, A_{f1}) - S^b(q, A_{f1}) = A - \sigma. \tag{A24}
\]

Equations (A22) and (A23) determine the segment of bid-ask midpoint and spread curves between \((A_{d1}, A_{f1})\), respectively.

**Case (c): if it turns out to be** \( b^c = M^c - S^c < \min[A - \sigma, -B + \sigma] \) **and** \( a^c = M^c + S^c > \max[A - \sigma, -B + \sigma] \) **(refer to Figures 11(c1) and 11(c2))**, then we can solve \( M^c \) and \( S^c \) by equations (A25) and (A26):

\[
(q)e^{\gamma[\rho - (\rho + 2)r]S^b}[e^{\gamma\rho(A - M^b)} + e^{-\gamma \rho(A - M^b)}] + (1 - q)[e^{-\gamma 2r S^b} + e^{-\gamma[(\rho + 2)r S^b \pm (M^b + B) - \rho \sigma]}] = 2^\alpha. \tag{A23}
\]

**Case (d): if it turns out to be** \( b^d = M^d - S^d \geq -B + \sigma \) **and** \( a^d = M^d + S^d > A - \sigma \), then we can solve \( M^d \) and \( S^d \) by equations (A27) and (A28):

\[
(q)e^{-\gamma[(\rho + 2)r S^d + \rho(M^d - A) - \rho \sigma]}[e^{-\gamma[(\rho + 2)r S^d + \rho(M^d - A) - \rho \sigma]} + e^{-\gamma 2r S^d}] + (1 - q)[e^{-\gamma 2r S^d} + e^{-\gamma[(\rho + 2)r S^d - \rho(M^d + B) - \rho \sigma]}] = 2^\alpha. \tag{A26}
\]

We solve the second liquidity dry-up threshold \( A_{d2} \) by the following equation (A29) given \( q \):

\[
M^d(q, A_{d2}) - S^d(q, A_{d2}) = -B + \sigma. \tag{A29}
\]

Equations (A27) and (A28) determine the segment of bid-ask midpoint and spread curves between \([A_{d2}, A_{f2})\), respectively.
Case (e): if it turns out to be $b^e = M^e - S^e \geq -B + \sigma$ and $a^e = M^e + S^e \leq A - \sigma$, then we can solve $M^e$ and $S^e$ by equations (A30) and (A31):

\[
(q)[e^{\gamma[(A-M^e)+\rho a]} + e^{\gamma[(A-M^e)-\rho a]}]^{\alpha} - (1-q)[e^{\gamma[(M^e+B)-\rho a]} + e^{\gamma[(M^e+B)+\rho a]}]^{\alpha} = 0,
\]

and

\[
e^{-\alpha\gamma(\rho+2r)S^e} \left\{ (q)[e^{\gamma[(A-M^e)+\rho a]} + e^{\gamma[(A-M^e)-\rho a]}]^{\alpha} + (1-q)[e^{\gamma[(M^e+B)-\rho a]} + e^{\gamma[(M^e+B)+\rho a]}]^{\alpha} \right\} = 2^\alpha.
\]

We solve the second liquidity flooding threshold $A_{f2}$ by the following equation (A32) given $q$:

\[
M^e(q, A_{f2}) + S^e(q, A_{f2}) = A - \sigma.
\]

Equations (A30) and (A31) determine the rightmost segment of bid-ask midpoint and spread curves, respectively.

To solve $A_{d1}$ and $\frac{dA_{d1}}{dq}$.

Plugging $A_{d1}$ into equation (A19) and taking derivative w.r.t. $q$ on both sides gives

\[
(\Psi_1^-)(\Psi_1^+)^{\alpha-1} - q\gamma\rho\left(\frac{dA_{d1}}{dq} - \frac{dM^a(A_{d1}, q)}{dq}\right)(\Psi_1^+)\alpha - q(\alpha - 1)\gamma\rho\left(\frac{dA_{d1}}{dq} - \frac{dM^a(A_{d1}, q)}{dq}\right)(\Psi_1^-)^2(\Psi_1^+)^{\alpha-2} - (\Psi_2^-)(\Psi_2^+)\alpha-1 + (1-q)\gamma\rho(\frac{dM^a(A_{d1}, q)}{dq})((\Psi_2^-)^2(\Psi_2^+)\alpha - 2 = 0,
\]

where $\Psi_1^- = -e^{\gamma\rho(A-M^e)} + e^{-\gamma\rho(A-M^e)}$, $\Psi_2^- = -e^{-\gamma\rho(B+M^e)} + e^{\gamma\rho(B+M^e)}$, $\Psi_1^+ = e^{\gamma\rho(A-M^e)} + e^{-\gamma\rho(A-M^e)}$, and $\Psi_2^+ = e^{\gamma\rho(B+M^e)} + e^{-\gamma\rho(B+M^e)}$.

Let $q = \frac{1}{2}$, the above equation can be reorganized as

\[
\gamma\rho\left(\frac{dM^a(A_{d1}, q)}{dq}\right)\left[\frac{1}{2}(\Psi_1^-)^2 + (\alpha - 1)(\Psi_1^-)^2(\Psi_1^+)^{\alpha-2}\right] + 2\gamma\rho A_{d1}\left[(\Psi_1^+)^\alpha + (\alpha - 1)(\Psi_1^-)^2(\Psi_1^+)^{\alpha-2}\right] = -2(\Psi_1^-)(\Psi_1^+)\alpha-1.
\]

Notice that $\frac{dA_{d1}}{dq} = 4A_{d1} + \frac{dA_{d1}}{dq}$.

Plugging $A_{d1}$ into equation (A20) and taking derivative w.r.t. $q$ on both sides gives

\[
-\alpha\gamma(\rho + 2r)\frac{dS^a(A_{d1}, q)}{dq}e^{\alpha\gamma[\rho\sigma-(\rho+2r)S^a]}\left[(q)(\Psi_1^+)^\alpha + (1-q)(\Psi_2^+)^\alpha\right] + e^{\alpha\gamma[\rho\sigma-(\rho+2r)S^a]}\left[(\Psi_1^+)^\alpha - q\alpha\gamma\rho\left(\frac{dA_{d1}}{dq} - \frac{dM^a(A_{d1}, q)}{dq}\right)(\Psi_1^-)(\Psi_1^+)\alpha-1
\]

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Let $q = \frac{1}{2}$, the above equation can be reorganized as
\[
(\rho + 2r) \frac{dS^a(A_{d1}, q = \frac{1}{2})}{dq} (\Psi_1^+) + \rho \frac{dA_{d1}}{dq} (\Psi_1^-) = -2 \rho A_{d1} (\Psi_1^-),
\]
or
\[
\frac{dS^a(A_{d1}, q = \frac{1}{2})}{dq} = -\rho \frac{\Psi_1^-}{\rho + 2r (\Psi_1^+)} (2A_{d1} + \frac{dA_{d1}}{dq}). \tag{A34}
\]
Plugging $A_{d1}$ into equation (A21) and taking derivative w.r.t. $q$ on both sides gives
\[
\frac{dM^a(A_{d1}, q)}{dq} + 2A_{d1} + \frac{dS^a(A_{d1}, q)}{dq} = -2A_{d1} - \frac{dA_{d1}}{dq}. \tag{A35}
\]
Combining equations (A34) and (A35) gives
\[
\frac{dM^a(A_{d1}, q = \frac{1}{2})}{dq} + 2A_{d1} = (-\rho \frac{\Psi_1^-}{\rho + 2r (\Psi_1^+)} - 1)(2A_{d1} + \frac{dA_{d1}}{dq}). \tag{A36}
\]
Combining equations (A36) and (A33) gives
\[
\gamma \rho (\frac{\rho}{\rho + 2r (\Psi_1^+)} - 1)(2A_{d1} + \frac{dA_{d1}}{dq})[(\Psi_1^+)^\alpha + (\alpha - 1)(\Psi_1^-)^2 (\Psi_1^+)^{\alpha-2}] = -2(\Psi^-_1)(\Psi_1^+)\alpha-1,
\]
which implies
\[
2A_{d1} + \frac{dA_{d1}}{dq} \bigg|_{q = \frac{1}{2}} < 0. \tag{A37}
\]
Inequality (A37) gives
\[
\frac{dA_{d1}}{dq} \bigg|_{q = \frac{1}{2}} < 0. \tag{A38}
\]
Combining inequality (A37) and (A34) gives
\[
\frac{dS^a(A_{d1}, q)}{dq} \bigg|_{q = \frac{1}{2}} < 0.
\]
Because we have proved $\frac{\partial M^a(A, q)}{\partial q} \bigg|_{q = \frac{1}{2}} < 0$ in Proposition (3) and $\frac{\partial M^a(A, q)}{\partial A} \bigg|_{q = \frac{1}{2}} = 0$ in Proposition (4), we have
\[
\frac{dM^a(A_{d1}, q)}{dq} \bigg|_{q = \frac{1}{2}} = \frac{\partial M^a(A_{d1}, q)}{\partial q} \bigg|_{q = \frac{1}{2}} + \frac{\partial M^a(A_{d1}, q)}{\partial A_{d1}} \bigg|_{q = \frac{1}{2}} \cdot \frac{dA_{d1}}{dq} \bigg|_{q = \frac{1}{2}} < 0.
\]
**To solve $A_{f1}$ and $\frac{dA_{f1}}{dq}$.**

Plugging $A_{f1}$ into equation (A24) and taking derivative w.r.t. $q$ on both sides gives
\[
\frac{dM^b(A_{f1}, q)}{dq} - \frac{dS^b(A_{f1}, q)}{dq} = \frac{dA_{f1}}{dq},
\]

or

\[
\frac{dM^b(A_{f1}, q)}{dq} + 2A_{f1} = \frac{dA_{f1}}{dq} + 2A_{f1} + \frac{dS^b(A_{f1}, q)}{dq}. \tag{A39}
\]

Plugging \(A_{f1}\) into equation (A23) and taking derivative w.r.t. \(q\) on both sides gives

\[
\begin{align*}
[\gamma - \gamma^+]^\alpha - q\alpha\gamma(\rho + 2r)\frac{dS^b(A_{f1}, q)}{dq}[\gamma - \gamma^+]^\alpha \\
+ q\alpha\gamma\rho\left( \frac{dA_{f1}}{dq} - \frac{dM^b(A_{f1}, q)}{dq} \right)[\gamma - \gamma^+]^\alpha \\
+ (1 - q)\alpha[\gamma - \gamma^+]^\alpha - \gamma \cdot 2r \frac{dS^b(A_{f1}, q)}{dq} \gamma \\
+ [-\gamma(\rho + 2r)\frac{dS^b(A_{f1}, q)}{dq} + \gamma\rho\left( \frac{dM^b(A_{f1}, q)}{dq} + \frac{dB(A_{f1})}{dq} \right)] [\gamma - \gamma^+]^\alpha \\
= 0,
\end{align*}
\]

where \(\gamma = e^{-2r}, \gamma^+ = e^{-\gamma([\rho + 2r]S^b + \rho(M^b - A_{f1}) - \rho \sigma)},\) and \(\gamma^+ = e^{-\gamma([\rho + 2r]S^b - \rho(M^b + B(A_{f1})) - \rho \sigma]}\).

Notice that \(M^b(A_{f1}, q) - S^b(A_{f1}, q) = A_{f1} - \sigma\).

Letting \(q = \frac{1}{2}\) and using equation (A39), the above equation can be reorganized as

\[
\rho\left( \frac{dM^b(A_{f1}, q = \frac{1}{2})}{dq} + 2A_{f1} \right) \gamma^+ = \left[ 2\rho \gamma^+ + 2r(\gamma - \gamma^+) \right] \frac{dS^b(A_{f1}, q = 1/2)}{dq}. \tag{A40}
\]

Notice that \(\frac{dB(A_{f1})}{dq} = 4A_{f1} + \frac{dA_{f1}}{dq}\).

Plugging \(A_{f1}\) into equation (A22) and taking derivative w.r.t. \(q\) on both sides gives

\[
\begin{align*}
[\gamma^+ - \gamma][\gamma - \gamma^+]^\alpha - q\alpha\gamma(\rho + 2r)\frac{dS^b(A_{f1}, q)}{dq}[\gamma - \gamma^+]^\alpha - \gamma\gamma^+ \cdot 2r \frac{dS^b(A_{f1}, q)}{dq} \gamma \\
+ q\alpha\gamma\rho\left( \frac{dA_{f1}}{dq} - \frac{dM^b(A_{f1}, q)}{dq} \right)[\gamma - \gamma^+]^\alpha \\
+ (1 - q)\alpha[\gamma - \gamma^+]^\alpha - \gamma \cdot 2r \frac{dS^b(A_{f1}, q)}{dq} \gamma \\
+ [-\gamma(\rho + 2r)\frac{dS^b(A_{f1}, q)}{dq} + \gamma\rho\left( \frac{dM^b(A_{f1}, q)}{dq} + \frac{dB(A_{f1})}{dq} \right)] [\gamma - \gamma^+]^\alpha \\
= 0.
\end{align*}
\]

Letting \(q = \frac{1}{2}\) and using equation (A39), the above equation can be reorganized as

\[
\Lambda[\gamma - \gamma^+]^\alpha - 2\frac{dS^b(A_{f1}, q = 1/2)}{dq} = \left[ \gamma^+ - \gamma \right][\gamma - \gamma^+]^\alpha + \gamma^+ [\gamma - \gamma^+]^\alpha, \tag{A41}
\]

55
where

\[ \Lambda = \alpha \gamma [(\mathcal{Y}_1^+)^2 - (\mathcal{Y}^-)^2] + \alpha \rho \mathcal{Y}_1^+ (\mathcal{Y}_1^+ - \mathcal{Y}^-) + \alpha (\rho + r) (\mathcal{Y}_1^+)^2 + 2r (\mathcal{Y}^-)^2 + (3\rho + \alpha r + 2r) \mathcal{Y}_1^+ \mathcal{Y}^- . \]

Notice that \( \mathcal{Y}_1^+ (q = \frac{1}{2}) > 0 \) and \( \mathcal{Y}^- (q = \frac{1}{2}) > 0 \), and that \( \mathcal{Y}_1^+ (q = \frac{1}{2}) > \mathcal{Y}^- (q = \frac{1}{2}) \), because \( M^b + S^b < A + \sigma \). Therefore we have

\[ \frac{dS^b(A_{f1}, q)}{dq} |_{q = \frac{1}{2}} > 0. \]  

(A42)

First taking partial derivative w.r.t. \( q \) on both sides of equation (A23) and then imposing \( q = \frac{1}{2} \) gives

\[ \alpha \gamma (\rho + 2r)[\mathcal{Y}^- + \mathcal{Y}_1^+]^{\alpha} \frac{\partial S^b(A_{f1}, q = \frac{1}{2})}{\partial q} + \alpha [\gamma \cdot 2r \mathcal{Y}^- + (\rho + 2r) \mathcal{Y}_1^+] [\mathcal{Y}^- + \mathcal{Y}_1^+]^{\alpha - 1} \frac{\partial S^b(A_{f1}, q = \frac{1}{2})}{\partial q} = \alpha \gamma \rho \mathcal{Y}^- [\mathcal{Y}^- + \mathcal{Y}_1^+]^{\alpha - 1} \frac{\partial M^b(A_{f1}, q = \frac{1}{2})}{\partial q} . \]

(A43)

Because \( \frac{dM^b(A_{f1}, q)}{dq} |_{q = \frac{1}{2}} = \frac{\partial M^b(A_{f1}, q)}{\partial q} |_{q = \frac{1}{2}} + \frac{\partial M^b(A_{f1}, q)}{\partial A_{f1}} |_{q = \frac{1}{2}} \cdot \frac{dA_{f1}}{dq} |_{q = \frac{1}{2}} \) and \( \frac{\partial M^b(A_{f1}, q)}{\partial A_{f1}} |_{q = \frac{1}{2}} = 0 \), we have

\[ \frac{dM^b(A_{f1}, q)}{dq} |_{q = \frac{1}{2}} = \frac{\partial M^b(A_{f1}, q)}{\partial q} |_{q = \frac{1}{2}} . \]  

(A44)

Because \( \frac{dS^b(A_{f1}, q)}{dq} |_{q = \frac{1}{2}} = \frac{\partial S^b(A_{f1}, q)}{\partial q} |_{q = \frac{1}{2}} + \frac{\partial S^b(A_{f1}, q)}{\partial A_{f1}} |_{q = \frac{1}{2}} \cdot \frac{dA_{f1}}{dq} |_{q = \frac{1}{2}} \), we have

\[ \frac{\partial S^b(A_{f1}, q)}{\partial q} |_{q = \frac{1}{2}} = \frac{dS^b(A_{f1}, q)}{dq} |_{q = \frac{1}{2}} - \frac{\partial S^b(A_{f1}, q)}{\partial A_{f1}} |_{q = \frac{1}{2}} \cdot \frac{dA_{f1}}{dq} |_{q = \frac{1}{2}} . \]  

(A45)

Combining equations (A44), (A45), and (A43) gives

\[ \Omega_1^+ \frac{dS^b(A_{f1}, q)}{dq} |_{q = \frac{1}{2}} = \Omega_2^+ \frac{dA_{f1}}{dq} |_{q = \frac{1}{2}} , \]  

(A46)

where

\[ \Omega_1^+ = \rho \mathcal{Y}_1^+ [\mathcal{Y}^- + \mathcal{Y}_1^+]^{\alpha - 1} + 2r [\mathcal{Y}^- + \mathcal{Y}_1^+]^{\alpha} + [2r \mathcal{Y}^- + (\rho + 2r) \mathcal{Y}_1^+] [\mathcal{Y}^- + \mathcal{Y}_1^+]^{\alpha - 1} > 0 , \]

and

\[ \Omega_2^+ = (\rho + 2r) [\mathcal{Y}^- + \mathcal{Y}_1^+]^{\alpha} \frac{\partial S^b(A_{f1}, q)}{\partial A_{f1}} |_{q = \frac{1}{2}} . \]
Because \( \frac{\partial S^b(A_{f1}, q)}{\partial A_{f1}} \big|_{q=\frac{1}{2}} > 0 \), equation (A46) implies

\[
\frac{dA_{f1}}{dq} \big|_{q=\frac{1}{2}} > 0. \tag{A47}
\]

To solve \( A_{d2} \) and \( \frac{dA_{d2}}{dq} \).

In a similar way of solving \( A_{f1} \), by considering case (d), we have

\[
\frac{dA_{d2}}{dq} \big|_{q=\frac{1}{2}} < 0. \tag{A48}
\]

To solve \( A_{f2} \) and \( \frac{dA_{f2}}{dq} \).

In a similar way of solving \( A_{d1} \), by considering case (e), we have

\[
\frac{dA_{f2}}{dq} \big|_{q=\frac{1}{2}} > 0. \tag{A49}
\]

To solve the jump around \( A_{d1} \).

Plugging \( A_{d1} \) into equations (A19) and (A22) and rearranging gives

\[
(q)[-e^{-\gamma \rho(A_{d1}-M^a)} + e^{-\gamma \rho(A_{d1}-M^a)}][e^{-\gamma \rho(A_{d1}-M^a)} + e^{-\gamma \rho(A_{d1}-M^a)}]^{\alpha-1} + (1-q)[-e^{-\gamma \rho(B(A_{d1})+M^b)} + e^{-\gamma \rho(B(A_{d1})+M^b)}][e^{-\gamma \rho(B(A_{d1})+M^b)} + e^{-\gamma \rho(B(A_{d1})+M^b)}]^{\alpha-1} = 0, \tag{A50}
\]

and

\[
(q)[-e^{-\gamma \rho(A_{d1}-M^b)} + e^{-\gamma \rho(A_{d1}-M^b)}][e^{-\gamma \rho(A_{d1}-M^b)} + e^{-\gamma \rho(A_{d1}-M^b)}]^{\alpha-1} + (1-q)e^{-\gamma \rho(B(A_{d1})+M^b)}[e^{-\gamma \rho(S^b-S)} + e^{-\gamma \rho(B(A_{d1})+M^b)}]^{\alpha-1} = 0. \tag{A51}
\]

Because \( e^{-\gamma \rho(B(A_{d1})+M^b)} > -e^{-\gamma \rho(B(A_{d1})+M^b)} + e^{-\gamma \rho(B(A_{d1})+M^b)} \) (because \( B(A_{d1}) + M^b > 0 \)) and \( e^{-\gamma \rho(S^b-S)} > e^{-\gamma \rho(B(A_{d1})+M^b)} \) (because \( M^b + S^b > -B + \sigma \)), comparing equation (A50) with equation (A51) gives

\[
M^a(A_{d1}, q) \leq M^b(A_{d1}, q), \forall q. \tag{A52}
\]

Plugging \( A_{d1} \) into equations (A23) gives

\[
(q)e^{\alpha \rho[\rho - (\rho+2r)S^b]}[e^{-\gamma \rho(A_{d1}-M^a)} + e^{-\gamma \rho(A_{d1}-M^b)}]^{\alpha} + (1-q)[e^{-\gamma \rho(S^b-S)} + e^{-\gamma [(\rho+2r)S^b - \rho(M^b+B(A_{d1}))-\rho]S^b}]^{\alpha} = 2^\alpha. \tag{A53}
\]
Rearranging equation (A53) yields
\[
(q)e^{\gamma[\rho-8(\rho+2\tau)S_b]}[e^{\gamma\rho(A_{d1}-M^b)} + e^{-\gamma\rho(A_{d1}-M^b)}]\alpha \\
+ (1 - q)[e^{-\gamma\rho(A_{d1}-M^b)} + e^{-\gamma\rho(A_{d1}-M^b)}]\alpha = e^{\alpha\gamma[\rho-8(\rho+2\tau)S_b]}\{\alpha\} [e^{\gamma\rho(A_{d1}-M^b)} + e^{-\gamma\rho(A_{d1}-M^b)}]\alpha \\
+ (1 - q)[e^\gamma(B(A_{d1})+M^b)]\alpha \quad \text{(Step 1.1)}
\]
\[
> e^{\alpha\gamma[\rho-8(\rho+2\tau)S_b]}\{\alpha\} [e^{\gamma\rho(A_{d1}-M^b)} + e^{-\gamma\rho(A_{d1}-M^b)}]\alpha \\
+ (1 - q)[e^{-\gamma\rho(B(A_{d1})+M^b)} + e^{-\gamma\rho(B(A_{d1})+M^b)}]\alpha \quad \text{(Step 1.2)}
\]
\[
> e^{\alpha\gamma[\rho-8(\rho+2\tau)S_b]}\{\alpha\} [e^{\gamma\rho(A_{d1}-M^a)} + e^{-\gamma\rho(A_{d1}-M^a)}]\alpha \\
+ (1 - q)[e^{-\gamma\rho(B(A_{d1})+M^a)} + e^{-\gamma\rho(B(A_{d1})+M^a)}]\alpha \quad \text{(Step 1.3)}
\]

Notice that Step 1.1 goes through to Step 1.2 because \(M^b + S^b > -B + \sigma\), and that Step 1.2 goes through to Step 1.3 because \(M^a\) is set to minimize \{\alpha\} [e^{\gamma\rho(A_{d1}-M^a)} + e^{-\gamma\rho(A_{d1}-M^a)}] + (1 - q)[e^{-\gamma\rho(B(A_{d1})+M^a)} + e^{-\gamma\rho(B(A_{d1})+M^a)}]\alpha\}. Therefore the above analysis shows
\[
S^a(A_{d1}, q) < S^b(A_{d1}, q), \forall q. \quad \text{(A54)}
\]

**To solve the jump around \(A_{f1}\).**

Given that \(M^b - S^b = A_{f1} - \sigma\), plugging \(A_{f1}\) into equations (A22) and rearranging gives
\[
(q)[e^{\gamma\rho(A_{f1}-M^b)} - e^{-\gamma\rho(A_{f1}-M^b)}] [e^{\gamma\rho(A_{f1}-M^b-S^b+\sigma)} + 1]^{\alpha-1} \\
- (1 - q)e^{\gamma\rho(A_{f1})[1 + e^{\gamma\rho(A_{f1})}]^{\alpha-1} = 0. \quad \text{(A55)}
\]

Plugging \(A_{f1}\) into equations (A25) and rearranging gives
\[
(q)[e^{\gamma\rho(A_{f1}-M^c)}] [e^{\gamma\rho(A_{f1}-M^c-S^c+\sigma)} + 1]^{\alpha-1} \\
- (1 - q)e^{\gamma\rho(A_{f1})[1 + e^{\gamma\rho(A_{f1})}]^{\alpha-1} = 0. \quad \text{(A56)}
\]

Given that \(M^b - S^b = A_{f1} - \sigma\), plugging \(A_{f1}\) into equations (A23) and rearranging gives
\[
(q)[e^{-\gamma[(\rho+2\tau)S^b+(M^b-A_{f1})-\rho\sigma]} + e^{-\gamma2rS^b}]^{\alpha} \\
+ (1 - q)[e^{-\gamma2rS^b} + e^{-\gamma[(\rho+2\tau)S^b+(M^b+B(A_{f1})-\rho\sigma)]} = 2^{\alpha}. \quad \text{(A57)}
\]

Plugging \(A_{f1}\) into equations (A26) and rearranging gives
\[
(q)[e^{-\gamma[(\rho+2\tau)S^c+(M^c-A_{f1})-\rho\sigma]} + e^{-\gamma2rS^c}]^{\alpha} \\
+ (1 - q)[e^{-\gamma2rS^c} + e^{-\gamma[(\rho+2\tau)S^c+(M^c+B(A_{f1})-\rho\sigma)]} = 2^{\alpha}. \quad \text{(A58)}
\]

We discover that equations (A57) and (A58) are in the same format. Because \(M^b \neq M^c\) and \(M^c\)
is chosen to minimize $S^c$ through equation (A58), we have

$$S^b(A_{f1}, q) > S^c(A_{f1}, q), \forall q. \quad (A59)$$

Because it is true that $M^b - S^b = A_{f1} - \sigma$ and $M^c - S^c < A_{f1} - \sigma$, we have

$$M^b(A_{f1}, q) > M^c(A_{f1}, q), \forall q. \quad (A60)$$

**To solve the jump around $A_{d2}$.**

In a similar way of solving the jump around $A_{f1}$, by comparing case (c) and case (d), we have

$$M^c(A_{d2}, q) < M^d(A_{d2}, q), \forall q. \quad (A61)$$

and

$$S^c(A_{d2}, q) < S^d(A_{d2}, q), \forall q. \quad (A62)$$

**To solve the jump around $A_{f2}$.**

In a similar way of solving the jump around $A_{d1}$, by comparing case (d) and case (e), we have

$$M^d(A_{f2}, q) > M^e(A_{f2}, q), \forall q. \quad (A63)$$

and

$$S^d(A_{f2}, q) > S^e(A_{f2}, q), \forall q. \quad (A64)$$

The proof is thus completed. ■
Online Appendix: Solving Equilibrium in the Dynamic Setting

We first introduce the following notations:

**Notations** (1) The beliefs on each world: \( \pi^t_G \) is the belief on the GOOD world at date \( t \), and \( \pi^t_B \) is the belief on the BAD world at date \( t \), where \( \pi^t_G + \pi^t_B = 1 \). (2) The beliefs on each state conditional on each world: \( \pi^t_{u|G} \) is the belief on the UP state conditional on the model of the world at date \( t \), and \( \pi^t_{d|G} \) is the belief on the DOWN state conditional on the model of the world at date \( t \), where \( \pi^t_{u|G} + \pi^t_{d|G} = 1 \), \( \pi^t_{u|B} + \pi^t_{d|B} = 1 \). (3) The beliefs on each state of each world: \( \pi^t_{u,G} \) is the subjective probability of the UP state happening in the GOOD world at date \( t \), and the same notation rule applies to \( \pi^t_{d,G} \), \( \pi^t_{u,B} \), and \( \pi^t_{d,B} \), where \( \pi^t_{u,G} + \pi^t_{d,G} + \pi^t_{u,B} + \pi^t_{d,B} = 1 \), \( \pi^t_{G} = \pi^t_{u,G} + \pi^t_{d,G} \), and \( \pi^t_{B} = \pi^t_{u,B} + \pi^t_{d,B} \). By Bayes’ rule\(^{35} \), \( \pi^t_{u,G} = \frac{\pi^t_{G}}{\pi^t_{G} \pi^t_{G}} \), \( \pi^t_{d,G} = \frac{\pi^t_{G}}{\pi^t_{G} \pi^t_{G}} \), \( \pi^t_{u,B} = \frac{\pi^t_{B}}{\pi^t_{B} \pi^t_{B}} \), and \( \pi^t_{d,B} = \frac{\pi^t_{B}}{\pi^t_{B} \pi^t_{B}} \). (4) The beliefs on the upcoming orders: \( \pi^t_i(b_t) \), \( \pi^t_i(a_t) \), and \( \pi^t_i(n_t) \) are the (subjective) probabilities of a buy, a sell, and a zero order given \( i \) (the state conditional on the world) respectively, \( i = u \cdot G, d \cdot G, u \cdot B, d \cdot B \). (5) The updating process of beliefs: by Bayesian Theorem, the beliefs at date \( t + 1 \) following a sell/buy/zero order at date \( t \) are given respectively by

\[
\begin{align*}
\pi^t_{i}(\text{sell order at } t) &= \frac{\pi^t_i \cdot \pi^t_{i}(b_t)}{\sum_i \pi^t_i \cdot \pi^t_{i}(b_t)}, \\
\pi^t_{i}(\text{buy order at } t) &= \frac{\pi^t_i \cdot \pi^t_{i}(a_t)}{\sum_i \pi^t_i \cdot \pi^t_{i}(a_t)}, \\
\pi^t_{i}(\text{zero order at } t) &= \frac{\pi^t_i \cdot \pi^t_{i}(n_t)}{\sum_i \pi^t_i \cdot \pi^t_{i}(n_t)},
\end{align*}
\]

where \( i = u \cdot G, d \cdot G, u \cdot B, \) and \( d \cdot B \). By assumption, the beliefs at date 1 are \( \pi^1_G = q \), \( \pi^1_B = 1 - q \), \( \pi^1_{u,G} = \pi^1_{d,G} = \pi^1_{u,B} = \pi^1_{d,B} = \frac{1}{2} \), \( \pi^1_{u,G} = \pi^1_{d,B} = \frac{1}{2} \), and \( \pi^1_{u,B} = \pi^1_{d,B} = \frac{1}{2}(1 - q) \).

Next we solve the general terms in the following six cases when the bid-ask range falling in different intervals.

**Case (a):** if it turns out to be \( b^*_t = M^*_t - S^*_t \geq A - \sigma \) and \( a^*_t = M^*_t + S^*_t \leq -B + \sigma \), then

\[
E[u(w)] = -\pi^t_{a|u} \exp\left\{-\gamma[(\rho + r)(M_t + S_t - \mu - \sigma) + r(\mu + \sigma - M_t + S_t)]\right\} - \pi^t_{d|a} \exp\left\{-\gamma[(\rho + r)(\mu - \sigma - M_t + S_t) + r(M_t + S_t - \mu + \sigma)]\right\} = -e^{\gamma(\rho - \rho + 2\gamma)S_t}|\pi^t_{u|u} e^{\gamma(\rho - \rho + 2\gamma)} + \pi^t_{d|d} e^{-\gamma(\rho - \rho + 2\gamma)}|.
\]

Therefore

\[
U = E[\phi(E[u(w)])]
\]

35Following Epstein and Schneider (2010) and Ju and Miao (2012), we assume ambiguity-averse market makers employ Bayes’ rule for belief updating.
impose

From equation (\(S_t\)), we have

\[g_t^{11}(M_t) = \pi^t_G \left[ \pi^t_u e^{\gamma (A-M_t)} + \pi^t_d e^{-\gamma A-M_t} \right] \alpha + \pi^t_B \left[ \pi^t_u e^{-\gamma (B+M_t)} + \pi^t_d e^{\gamma (B+M_t)} \right] \alpha \]

and impose

\[
\frac{1}{\alpha \gamma \rho} \frac{\partial g_t^{11}}{\partial M_t} = \pi^t_G \left[ \pi^t_u e^{\gamma (A-M_t)} + \pi^t_d e^{-\gamma A-M_t} \right] \alpha - 1 + \pi^t_B \left[ \pi^t_u e^{-\gamma (B+M_t)} + \pi^t_d e^{\gamma (B+M_t)} \right] \alpha - 1 = 0. \tag{B2}
\]

From equation (B2), we can solve \(M_t^*\).

From equation (B1), we have

\[e^{\alpha [\rho (\sigma-(\rho+2)S_t)]} \left[ \pi^t_G \left[ \pi^t_u e^{\gamma (A-M_t^*)} + \pi^t_d e^{-\gamma A-M_t^*} \right] \alpha + \pi^t_B \left[ \pi^t_u e^{-\gamma (B+M_t^*)} + \pi^t_d e^{\gamma (B+M_t^*)} \right] \alpha \right] = 1. \tag{B3}
\]

From equation (B3), we can solve \(S_t^*\).

If Case (a) turns out to be the (Nash) equilibrium, then the beliefs are

\[
\begin{align*}
\pi^t_u_G(b_t) &= r, & \pi^t_u_G(a_t) &= \rho + r, & \pi^t_u_G(n_t) &= 1 - \rho - 2r, \\pi^t_d_G(b_t) &= \rho + r, & \pi^t_d_G(a_t) &= r, & \pi^t_d_G(n_t) &= 1 - \rho - 2r, \\pi^t_u_B(b_t) &= r, & \pi^t_u_B(a_t) &= \rho + r, & \pi^t_u_B(n_t) &= 1 - \rho - 2r, \\pi^t_d_B(b_t) &= \rho + r, & \pi^t_d_B(a_t) &= r, & \pi^t_d_B(n_t) &= 1 - \rho - 2r.
\end{align*}
\]

Case (b): if it turns out to be \(b_t^* = M_t^* - S_t^* \geq -B + \sigma\) and \(a_t^* = M_t^* + S_t^* \leq A + \sigma\), then

\[
E[u(w)|\text{GOOD world}] = -\pi^t_u e^{\gamma [\rho (\sigma-(\rho+2)S_t)]} \left[ \pi^t_u e^{\gamma (A-M_t)} + \pi^t_d e^{-\gamma A-M_t} \right] + \pi^t_B e^{\gamma [\rho (M_t-A) + \rho \sigma]} \right],
\]

and

\[
E[u(w)|\text{BAD world}] = -\pi^t_u e^{\gamma [\rho (\sigma-(\rho+2)S_t)]} \left[ \pi^t_u e^{\gamma (B-M_t)} + \pi^t_d e^{-\gamma (B-M_t)} \right] + \pi^t_B e^{\gamma [\rho (M_t-B) + \rho \sigma]} \right].
\]
Therefore

\[ U = E[\phi(E[u(w)])] \]

\[ = -\pi_t^t \frac{1}{\alpha} \left( e^{-\alpha \gamma (\rho + 2 \tau) S_t} \left[ \pi_u^t e^{-\gamma \rho (M_t - A) - \rho \sigma} + \pi_d^t e^{\gamma (\rho (M_t - A) + \rho \sigma)} \right] \right. \]

\[ - \pi_B^t \frac{1}{\alpha} e^{-\alpha \gamma (\rho + 2 \tau) S_t} \left[ \pi_u^t e^{\gamma \rho (M_t - B) - \rho \sigma} + \pi_d^t e^{\gamma (\rho (M_t - B) + \rho \sigma)} \right] \]

\[ = -\frac{1}{\alpha}, \]

or

\[ e^{-\alpha \gamma (\rho + 2 \tau) S_t} \left\{ \pi_G^t [\pi_u^t e^{-\gamma \rho (M_t - A) - \rho \sigma}] + \pi_d^t e^{\gamma (\rho (M_t - A) + \rho \sigma)} \right\} + \pi_B^t [\pi_u^t e^{\gamma \rho (M_t - B) - \rho \sigma}] + \pi_d^t e^{\gamma (\rho (M_t - B) + \rho \sigma)} \right\} = 1. \]  

(B4)

Let \( g_t^{1,2}(M_t) = \pi_G^t [\pi_u^t e^{-\gamma \rho (M_t - A) - \rho \sigma}] + \pi_d^t e^{\gamma (\rho (M_t - A) + \rho \sigma)} \right\} + \pi_B^t [\pi_u^t e^{\gamma \rho (M_t - B) - \rho \sigma}] + \pi_d^t e^{\gamma (\rho (M_t - B) + \rho \sigma)} \right\} = 1. \) 

Then impose

\[ \frac{1}{\alpha \gamma \rho} \frac{\partial g_t^{1,2}}{\partial M_t} = \pi_G^t [\pi_u^t e^{-\gamma \rho (M_t - A) - \rho \sigma}] + \pi_d^t e^{\gamma (\rho (M_t - A) + \rho \sigma)} \right\} + \pi_B^t [\pi_u^t e^{\gamma \rho (M_t - B) - \rho \sigma}] + \pi_d^t e^{\gamma (\rho (M_t - B) + \rho \sigma)} \right\} = 0. \]  

(B5)

From equation (B5), we can solve \( M_t^* \).

From equation (B4), we have

\[ e^{-\alpha \gamma (\rho + 2 \tau) S_t} \left\{ \pi_G^t [\pi_u^t e^{-\gamma \rho (M_t^* - A) - \rho \sigma}] + \pi_d^t e^{\gamma (\rho (M_t^* - A) + \rho \sigma)} \right\} + \pi_B^t [\pi_u^t e^{\gamma \rho (M_t^* - B) - \rho \sigma}] + \pi_d^t e^{\gamma (\rho (M_t^* - B) + \rho \sigma)} \right\} = 1. \]  

(B6)

From equation (B6), we can solve \( S_t^* \).

If Case (b) turns out to be the (Nash) equilibrium, then the beliefs are

\[ \pi_{u,G}(b_t) = r, \quad \pi_{d,G}(a_t) = \rho + r, \quad \pi_{u,G}(n_t) = 1 - \rho - 2r, \]

\[ \pi_{d,G}(b_t) = r + \rho, \quad \pi_{d,G}(a_t) = r, \quad \pi_{d,G}(n_t) = 1 - \rho - 2r, \]

\[ \pi_{u,B}(b_t) = r + \rho, \quad \pi_{u,B}(a_t) = r, \quad \pi_{u,B}(n_t) = 1 - \rho - 2r, \]

\[ \pi_{d,B}(b_t) = r + \rho, \quad \pi_{d,B}(a_t) = r, \quad \pi_{d,B}(n_t) = 1 - \rho - 2r. \]

Case (c): if it turns out to be \( b_t^* - S_t^* \geq -B - \sigma \) and \( a_t^* = M_t^* + S_t^* \leq A - \sigma \), then

\[ E[u(w)|\text{GOOD world}] = -\pi_u^t \exp\{-\gamma[(\rho + r)(M_t + S_t - A - \sigma) + (r)(A + \sigma - M_t + S_t)]\} \]
\[ -\pi_t^{d|G} \exp\{-\gamma[(\rho + r)(M_t + S_t - A + \sigma) + r(A - \sigma - M_t + S_t)]\} = - e^{-\gamma(\rho + 2r)}S_t\left[\pi_t^{u|G} e^{-\gamma[\rho(M_t - A) - \rho \sigma]} + \pi_t^{d|G} e^{-\gamma[\rho(M_t - A) + \rho \sigma]}\right], \]

and

\[ E[u(w) | \text{BAD world}] = -\pi_t^{u|B} \exp\{-\gamma[(\rho + r)(M_t + S_t + B - \sigma) + r(-B + \sigma - M_t + S_t)]\} - e^{-\gamma(\rho + 2r)}S_t\left[\pi_t^{u|B} e^{-\gamma[\rho(M_t + B) - \rho \sigma]} + \pi_t^{d|B} e^{-\gamma[\rho(M_t + B) + \rho \sigma]}\right]. \]

Therefore

\[
U = E[\phi(E[u(w)])] = -\pi_t^{u|B} e^{-\alpha \gamma(\rho + 2r)}S_t\left[\pi_t^{u|G} e^{-\gamma[\rho(M_t - A) - \rho \sigma]} + \pi_t^{d|G} e^{-\gamma[\rho(M_t - A) + \rho \sigma]}\right]^{\alpha} - \pi_t^{u|B} e^{-\alpha \gamma(\rho + 2r)}S_t\left[\pi_t^{u|B} e^{-\gamma[\rho(M_t + B) - \rho \sigma]} + \pi_t^{d|B} e^{-\gamma[\rho(M_t + B) + \rho \sigma]}\right]^{\alpha} = -\frac{1}{\alpha}. \tag{B7} \]

Let

\[ g_t^{1,3}(M_t) = \pi_t^{u|G} e^{-\gamma[\rho(M_t - A) - \rho \sigma]} + \pi_t^{d|G} e^{-\gamma[\rho(M_t - A) + \rho \sigma]} \]

\[ + \pi_t^{u|B} e^{-\gamma[\rho(M_t + B) - \rho \sigma]} + \pi_t^{d|B} e^{-\gamma[\rho(M_t + B) + \rho \sigma]} \]

then impose

\[
\frac{1}{\alpha \gamma \rho} \frac{\partial g_t^{1,3}}{\partial M_t} = \pi_t^{u|G}[-\pi_t^{u|G} e^{-\gamma[\rho(M_t - A) - \rho \sigma]} - \pi_t^{d|G} e^{-\gamma[\rho(M_t - A) + \rho \sigma]}] \]

\[ + \pi_t^{u|G}[-\pi_t^{u|G} e^{-\gamma[\rho(M_t - A) - \rho \sigma]} + \pi_t^{d|G} e^{-\gamma[\rho(M_t - A) + \rho \sigma]}\] \]

\[ + \pi_t^{u|B} e^{-\gamma[\rho(M_t + B) - \rho \sigma]} + \pi_t^{d|B} e^{-\gamma[\rho(M_t + B) + \rho \sigma]}\] \]

\[ = 0. \tag{B8} \]

From equation (B8), we can solve \( M_t^* \).

From equation (B7), we have

\[ e^{-\alpha \gamma(\rho + 2r)}S_t\left[\pi_t^{u|G} e^{-\gamma[\rho(M_t^* - A) - \rho \sigma]} + \pi_t^{d|G} e^{-\gamma[\rho(M_t^* - A) + \rho \sigma]}\right]^{\alpha} \]
Therefore

\[ + \pi^t_B [\pi^t_{u|B} e^{-\gamma (\rho (M_t^* + B) - \rho \sigma) + \pi^t_{d|B} e^{\gamma (\rho (M_t^* + B) + \rho \sigma)}]} ] = 1. \] (B9)

From equation (B9), we can solve \( S_t^* \).

If Case (c) turns out to be the (Nash) equilibrium, then the beliefs are

\[
\begin{align*}
\pi^t_{u,G}(b_t) &= r, & \pi^t_{u,G}(a_t) &= \rho + r, & \pi^t_{u,G}(n_t) &= 1 - \rho - 2r, \\
\pi^t_{d,G}(b_t) &= r, & \pi^t_{d,G}(a_t) &= \rho + r, & \pi^t_{d,G}(n_t) &= 1 - \rho - 2r, \\
\pi^t_{u,B}(b_t) &= r, & \pi^t_{u,B}(a_t) &= \rho + r, & \pi^t_{u,B}(n_t) &= 1 - \rho - 2r, \\
\pi^t_{d,B}(b_t) &= \rho + r, & \pi^t_{d,B}(a_t) &= r, & \pi^t_{d,B}(n_t) &= 1 - \rho - 2r.
\end{align*}
\]

Case (d): if it turns out to be \( b_t^* = M_t^* - S_t^* \in [A - \sigma, -B + \sigma] \) and \( a_t^* = M_t^* + S_t^* \in (-B + \sigma, A + \sigma] \), then

\[
\begin{align*}
E[u(w)|\text{GOOD world}] &= - \pi^t_{u|G} \exp \{-\gamma (\rho + r)(M_t + S_t + A - \sigma) + (r)(A + \sigma - M_t + S_t)]
\end{align*}
\]

\[
- \pi^t_{d|G} \exp \{-\gamma [(\rho + r)(A - \sigma - M_t + S_t) + (r)(M_t + S_t - A + \sigma)]
\end{align*}
\]

\[
= - \pi^t_{u|G} \exp \{-\gamma [(\rho + 2r)S_t + \rho (M_t - A) - \rho \sigma)]
\end{align*}
\]

\[
- \pi^t_{d|G} \exp \{-\gamma [(\rho + 2r)S_t - \rho (M_t - A) - \rho \sigma])
\end{align*}
\]

and

\[
\begin{align*}
E[u(w)|\text{BAD world}] &= - \pi^t_{u|B} \exp \{-\gamma [(\rho + r)(M_t + S_t + B - \sigma) + (r)(-B + \sigma - M_t + S_t)]
\end{align*}
\]

\[
- \pi^t_{d|B} \exp \{-\gamma [(\rho + r)(-B - \sigma - M_t + S_t) + (r)(M_t + S_t + B + \sigma)]
\end{align*}
\]

\[
= - \pi^t_{u|B} \exp \{-\gamma \cdot 2r S_t\} - \pi^t_{d|B} \exp \{-\gamma [(\rho + 2r)S_t - \rho (M_t + B) - \rho \sigma])
\end{align*}
\]

Therefore

\[
U = \mathbb{E}[\phi(E[u(w)])]) \equiv g^{1,4}_t (M_t, S_t)
\]

\[
= - \pi^t_G \frac{1}{\alpha} e^{\alpha [\rho - (\rho + 2r)S_t] \gamma (\rho (A - M_t) + \pi^t_{d|G} e^{-\gamma (\rho (A - M_t) + \pi^t_{d|B} e^{-\gamma (\rho (M_t + B) - \rho \sigma))})] \alpha
\]

\[
= - \frac{1}{\alpha}
\]

or

\[
\pi^t_G e^{\alpha [\rho - (\rho + 2r)S_t] \gamma (\rho (A - M_t) + \pi^t_{d|G} e^{-\gamma (\rho (A - M_t)] \alpha
\]

\[
+ \pi^t_B [\pi^t_{u|B} e^{-\gamma (\rho (A - M_t) - \rho (M_t + B) - \rho \sigma))]} \alpha = 1. \] (B10)

Therefore

\[
\frac{1}{\gamma \rho} \frac{\partial g^{1,4}_t}{\partial M_t} = \pi^t_G e^{\alpha [\rho - (\rho + 2r)S_t] \gamma (\rho (A - M_t) - \pi^t_{d|G} e^{-\gamma (\rho (A - M_t)] \alpha
\]

\[
+ \pi^t_B [\pi^t_{u|B} e^{-\gamma (\rho (A - M_t) - \rho (M_t + B) - \rho \sigma)))] \alpha = 1. \]
Combining equation \((B10)\) and equation \((B11)\), we can solve \(M_t^*\) and \(S_t^*\).

If Case (d) turns out to be the (Nash) equilibrium, then the beliefs are
\[
\begin{align*}
\pi^t_{u-G}(b_t) &= r, & \pi^t_{u-G}(a_t) &= \rho + r, & \pi^t_{u-G}(n_t) &= 1 - \rho - 2r, \\
\pi^t_{d-G}(b_t) &= \rho + r, & \pi^t_{d-G}(a_t) &= r, & \pi^t_{d-G}(n_t) &= 1 - \rho - 2r, \\
\pi^t_{u-B}(b_t) &= r, & \pi^t_{u-B}(a_t) &= r, & \pi^t_{u-B}(n_t) &= 1 - 2r, \\
\pi^t_{d-B}(b_t) &= \rho + r, & \pi^t_{d-B}(a_t) &= r, & \pi^t_{d-B}(n_t) &= 1 - \rho - 2r.
\end{align*}
\]

Case (e): if it turns out to be \(b_t^* = M_t^* - S_t^* \in [-B - \sigma, A - \sigma]\) and \(a_t^* = M_t^* + S_t^* \in (A - \sigma, -B + \sigma]\), then
\[
E[u(w)|\text{GOOD world}] = -\pi^t_{u-G}\exp\{-\gamma[(\rho + r)(M_t + S_t - A - \sigma) + (r)(A + \sigma - M_t + S_t)]\} \\
-\pi^t_{d-G}\exp\{-\gamma[(r)(A - \sigma - M_t + S_t) + (r)(M_t + S_t - A + \sigma)]\} \\
= -\pi^t_{u-G}\exp\{-\gamma[(\rho + 2r)S_t + \rho(M_t - A) - \rho\sigma]\} - \pi^t_{d-G}\exp\{-\gamma \cdot 2r S_t\},
\]
and
\[
E[u(w)|\text{BAD world}] = -\pi^t_{u-B}\exp\{-\gamma[(\rho + r)(M_t + S_t + B - \sigma) + (r)(-B + \sigma - M_t + S_t)]\} \\
-\pi^t_{d-B}\exp\{-\gamma[(\rho + r)(-B - \sigma - M_t + S_t) + (r)(M_t + S_t + B + \sigma)]\} \\
= -\pi^t_{u-B}\exp\{-\gamma[(\rho + 2r)S_t + \rho(M_t + B) - \rho\sigma]\} \\
- \pi^t_{d-B}\exp\{-\gamma[(\rho + 2r)S_t - \rho(M_t + B) - \rho\sigma]\}.
\]

Therefore
\[
U = E[\phi(E[u(w)])] \equiv q_t^{1.5}(M_t, S_t)
\]
Combining equations (B13) and (B12), we can solve $M_t^*$ and $S_t^*$. If Case (e) turns out to be the (Nash) equilibrium, then the beliefs are

\[
\begin{align*}
\pi_{u,G}(b_t) &= r, & \pi_{u,G}(a_t) &= \rho + r, & \pi_{u,G}(n_t) &= 1 - \rho - 2r, \\
\pi_{d,G}(b_t) &= r, & \pi_{d,G}(a_t) &= r, & \pi_{d,G}(n_t) &= 1 - 2r, \\
\pi_{u,B}(b_t) &= r, & \pi_{u,B}(a_t) &= \rho + r, & \pi_{u,B}(n_t) &= 1 - \rho - 2r, \\
\pi_{d,B}(b_t) &= \rho + r, & \pi_{d,B}(a_t) &= r, & \pi_{d,B}(n_t) &= 1 - \rho - 2r.
\end{align*}
\]
Case (f): if it turns out to be \( b_i^* = M_i^* - S_i^* < A - \sigma \) and \( a_i^* = M_i^* + S_i^* > -B + \sigma \), then

\[
E[u(w)|\text{GOOD world}] = -\pi_{uG}^t \exp\{-\gamma[(\rho + r)(M_t + S_t - A - \sigma) + (r)(A + \sigma - M_t + S_t)]\}
- \pi_{dG}^t \exp\{-\gamma[(r)(A - \sigma - M_t + S_t) + (r)(M_t + S_t - A + \sigma)]\}
= -\pi_{uG}^t \exp\{-\gamma[(\rho + 2r)S_t + \rho(M_t - A) - \rho\sigma]\} \pi_{dG}^t \exp\{-\gamma \cdot 2rS_t\},
\]

and

\[
E[u(w)|\text{BAD world}] = -\pi_{uB}^t \exp\{-\gamma[(r)(M_t + S_t + A - \sigma) + (r)(-A + \sigma - M_t + S_t)]\}
- \pi_{dB}^t \exp\{-\gamma[(\rho + r)(-A - \sigma - M_t + S_t) + (r)(M_t + S_t + A + \sigma)]\}
= -\pi_{uB}^t \exp\{-\gamma \cdot 2rS_t\} \pi_{dB}^t \exp\{-\gamma[(\rho + 2r)S_t - \rho(M_t + B) - \rho\sigma]\}.
\]

Therefore

\[
U = E[\phi(E[u(w)])] \equiv \eta_{16}^t(M_t, S_t)
= -\pi_{uG}^t \frac{1}{\alpha} [\pi_{uG}^t e^{-\gamma[(\rho + 2r)S_t + \rho(M_t - A) - \rho\sigma]} + \pi_{dG}^t e^{-\gamma \cdot 2rS_t}]^\alpha
- \pi_{dB}^t \frac{1}{\alpha} [\pi_{uB}^t e^{-\gamma 2rS_t} + \pi_{dB}^t e^{-\gamma[(\rho + 2r)S_t - \rho(M_t + B) - \rho\sigma]}]^\alpha
= -\frac{1}{\alpha},
\]

or

\[
\pi_{uG}^t [\pi_{uG}^t e^{-\gamma[(\rho + 2r)S_t + \rho(M_t - A) - \rho\sigma]} + \pi_{dG}^t e^{-\gamma \cdot 2rS_t}]^\alpha
+ \pi_{dB}^t [\pi_{uB}^t e^{-\gamma 2rS_t} + \pi_{dB}^t e^{-\gamma[(\rho + 2r)S_t - \rho(M_t + B) - \rho\sigma]}]^\alpha = 1.
\]

Therefore

\[
\frac{1}{\gamma \rho} \frac{\partial \eta_{16}^t}{\partial M_t} = \pi_{uG}^t [\pi_{uG}^t e^{-\gamma[(\rho + 2r)S_t + \rho(M_t - A) - \rho\sigma]} [\pi_{uG}^t e^{-\gamma[(\rho + 2r)S_t + \rho(M_t - A) - \rho\sigma]} + \pi_{dG}^t e^{-\gamma \cdot 2rS_t}]^{\alpha - 1}
- \pi_{dB}^t [\pi_{uB}^t e^{-\gamma[(\rho + 2r)S_t - \rho(M_t + B) - \rho\sigma]} [\pi_{uB}^t e^{-\gamma 2rS_t} + \pi_{dB}^t e^{-\gamma[(\rho + 2r)S_t - \rho(M_t + B) - \rho\sigma]}]^{\alpha - 1},
\]

(B14)

and

\[
\frac{\partial \eta_{16}^t}{\partial S_t} = \pi_{uG}^t [\gamma(\rho + 2r)\pi_{uG}^t e^{-\gamma[(\rho + 2r)S_t + \rho(M_t - A) - \rho\sigma]} + (\gamma \cdot 2r)\pi_{dG}^t e^{-\gamma \cdot 2rS_t}]
+ \pi_{dB}^t [\gamma(\rho + 2r)\pi_{dB}^t e^{-\gamma 2rS_t} + (\gamma \cdot 2r)\pi_{dB}^t e^{-\gamma[(\rho + 2r)S_t - \rho(M_t + B) - \rho\sigma]}[\pi_{uB}^t e^{-\gamma 2rS_t} + \pi_{dB}^t e^{-\gamma[(\rho + 2r)S_t - \rho(M_t + B) - \rho\sigma]}]^{\alpha - 1},
\]

\[> 0.\]
Let $\frac{\partial S_t}{\partial M_t} = -\frac{\partial_{t}^1}{\partial_{s}^1} = 0$, so we have $\frac{1}{\gamma p} \frac{\partial_{t}^1}{\partial_{s}^1} = 0$, or

\[
\pi^t_u G \pi^t_G e^{-\gamma \rho M_t} [\pi^t_u G e^{-\gamma [pS_t + \rho(M_t - A) - \rho]} + \pi^t_d G]^{\alpha - 1} - \pi^t_u B \pi^t_B e^{\gamma \rho M_t} [\pi^t_u B + \pi^t_d B e^{-\gamma [pS_t - \rho(M_t + B) - \rho]}]^{\alpha - 1} = 0.
\] (B15)

Combining equations (B15) and (B14), we can solve $M^*_t$ and $S^*_t$.

If Case (f) turns out to be the (Nash) equilibrium, then the beliefs are

\[
\begin{align*}
\pi^t_{uG}(b_t) &= r, & \pi^t_{uG}(a_t) &= \rho + r, & \pi^t_{uG}(n_t) &= 1 - \rho - 2r, \\
\pi^t_{dG}(b_t) &= r, & \pi^t_{dG}(a_t) &= r, & \pi^t_{dG}(n_t) &= 1 - 2r, \\
\pi^t_{uB}(b_t) &= r, & \pi^t_{uB}(a_t) &= r, & \pi^t_{uB}(n_t) &= 1 - 2r, \\
\pi^t_{dB}(b_t) &= \rho + r, & \pi^t_{dB}(a_t) &= r, & \pi^t_{dB}(n_t) &= 1 - \rho - 2r.
\end{align*}
\]