Production and Hedging with Optimism and Pessimism under Ambiguity

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Abstract: Firms are generally uncertain about the true probability distribution governing output prices and thus hold multiple subjective probabilistic beliefs about it. This paper analyzes the optimal production and hedging decisions of a competitive firm that holds optimism and pessimism under such price ambiguity. We first show that the separation theorem holds in that the firm’s optimal output level depends neither on the output price distribution nor on the firm’s preferences (including the risk attitude, the degree of ambiguity, and the optimism and pessimism levels of the firm under price ambiguity). Moreover, we show that whether the firm chooses a full hedge depends on the extent to which the firm is optimistic about the output price distribution, i.e., the validity of the full-hedging theorem depends on the firm’s optimism level under price ambiguity. The more optimistic the firm is under price ambiguity, the less likely the firm is to choose a full hedge. Most notably, while conventional wisdom holds that full-hedging is most reasonable when unbiased hedging derivatives markets exist, we identify a threshold of the firm’s optimism level above which full-hedging is never an optimal choice for the firm even when there exist unbiased forwards. Our results thus identify a novel circumstance under which the full-hedging theorem fails and shed light on the roles of optimism and pessimism under price ambiguity in a firm's output and hedging choices.

Keywords: Production, Hedging, Price Ambiguity, Optimism, Pessimism.
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1 INTRODUCTION

Joint production and hedging decisions are important and ubiquitous for firms because firms are generally faced with uncertainty about output prices and hence motivated to enter a hedge market. These tasks become particularly challenging when firms do not have enough information or observations to assess with precision the uncertainty they face and feel ambiguous about the true probability distribution governing output prices. When firms face such price ambiguity and cannot come up with a precise probability distribution of output prices, they may exhibit optimism (pessimism) by assigning more weight to favorable (unfavorable) probabilistic beliefs. How will firms make their output and hedging choices under price ambiguity? Will the production and hedging decisions of firms be affected by their optimistic and pessimistic attitudes towards price ambiguity? If so, how? The objective of this paper is to address these questions by considering a competitive firm of Sandmo (1970) within the context of the Choquet expected utility with neo-additive capacities (Chateauneuf et al., 2007). We show that, while the optimal output level for the firm is not affected by price ambiguity, the firm is more (less) likely to choose a full hedge when the firm’s degree of ambiguity (level of optimism) is higher. Especially, we identify a threshold of the firm’s optimism level above which the firm will never chooses a full-hedge. This paper thus sheds light on the roles of optimism and pessimism under price ambiguity in firms’ production and hedging decisions.

The standard theory of firm production and hedging under price uncertainty assumes that firms are expected utility maximizers, in conformity with the conventional von Neumann-Morgenstern-Savage expected utility paradigm. There are two well-celebrated results in this literature. The first one is the full-hedging theorem, stating that the firm should fully hedge

1Knight (1921) distinguishes two main sources of uncertainty: risk and ambiguity. While risk refers to uncertain outcomes with a specified probability distribution, ambiguity or Knightian Uncertainty refers to uncertain outcomes with unspecified or a multitude of probability distributions.

2Baron (1970) and Sandmo (1971) develop the pioneering models of optimal production under price uncertainty, which are extended by Danthine (1978), Holthausen (1979), and Feder et al. (1980) to incorporate optimal hedging decisions as well. Following these seminal papers, there has been a great volume of research on the production and hedging decisions of the competitive firm under price uncertainty (see, among many others, Benninga, Eldor, & Zilcha, 1984; Broll & Zilcha, 1992; Battermann et al., 2000; Akron & Benninga, 2013). Note that the expected utility theory encompasses the objective expected utility theory (von Neumann & Morgenstern, 1944) and the subjective expected utility theory (Savage, 1954).
the price risk if unbiased hedging derivatives markets exist. The second one is the separation theorem, stating that the firm’s production decision depends neither on the preferences of the firm nor on the probability distribution of output prices.\(^3\) Notwithstanding its primacy, experimental research has consistently showed that decision-makers may not behave as predicted by the expected utility paradigm.\(^4\) This casts doubt on the robustness of the full-hedging theorem and the separation theorem.

This study departs from the conventional expected utility framework and analyzes how a competitive firm’s optimism and pessimism under price ambiguity affect its production and hedging decisions, thereby re-examining the validity of the separation theorem and the full-hedging theorem. Starting with the paradox of Ellsberg (1961), the influence of ambiguity on decision-making has been well-documented (e.g., Einhorn & Hogarth, 1986; Sarin & Weber, 1993; Kilka & Weber, 2001). There is a growing body of research that explores how a competitive firm chooses its output level and hedging position when it is uncertain about the true probability distribution governing output prices. Lien (2000) adopts the maxmin expected utility model pioneered by Gilboa and Schmeidler (1989) to analyze firm production and hedging decision under price ambiguity. Lien and Wang (2003) extend the analysis of Lien (2000) to the futures market equilibrium. In contrast, Iwaki and Osaki (2012) and Wong (2015) adopt the smooth ambiguity aversion model developed by Klibanoff et al. (2005) to examine the production and hedging behavior of a competitive firm. These prior studies, however, do not take into account the roles of optimistic and pessimistic attitudes towards price ambiguity.

This paper complements the extant literature of firms’ production and hedging decisions by incorporating the concepts of optimism and pessimism under price ambiguity. To this end, we consider the competitive firm model of Sandmo (1970) within the context of the Choquet expected utility with neo-additive capacities (Chateauneuf et al., 2007).\(^5\) In a typical model of the Choquet expected utility with neo-additive capacities, an decision-maker makes decisions

\(^3\)The preferences of the firm are actually referred to the preferences of the firm’s decision-maker. Following the literature, we do not explicitly distinguish them in the context.


\(^5\)Models of the Choquet expected utility with neo-additive capacities are the generalization of expected utility that can accommodate optimistic and pessimistic attitudes towards ambiguity and have been proven useful in many applications (e.g., Teitelbaum, 2007; Schroder, 2011; Chakravarty & Kelsey, 2012; Ford, Kelsey, & Pang, 2013). Alternative approaches include (i) maxmin expected utility models (Gilboa & Schmeidler, 1989; Lien, 2000), where the decision-maker is uncertain about the underlying distribution and will assume the worst outcome while evaluating the payoff generated by his or her decision; and (ii) smooth ambiguity aversion models, where the decision-maker maximizes the second-order expectation of a concave transformation of the expected utility with respect to the second-order subjective beliefs (Klibanoff, Marinacci, & Mukerji, 2005; Wong, 2015). These alternative models serve their own purposes but do not accommodate optimism and pessimism under ambiguity.
under uncertainty as if he or she believes, with incomplete confidence, that a specified probability distribution describes the likelihood of certain events and chooses an act from the set of available acts that maximizes a weighted sum of the minimum utility, the maximum utility, and the expected utility with respect to such a probability distribution. The parameters of the model can capture the firm’s degree of ambiguity, confidence, optimism, and pessimism. The confidence and optimism concepts in the model also echo with the competence effect of Heath and Tversky (1991), i.e., people’s willingness to act on their own judgments is affected by their subjective competence.

Our main results are as follows. First, we show that the optimal output choice for the firm is independent of the output price distribution and the firm’s preferences (including the risk attitude, the degree of ambiguity, and the optimism and pessimism levels of the firm under price ambiguity). That is, the separation theorem holds within the context of the Choquet expected utility with neo-additive capacities (Chateauneuf et al., 2007). Moreover, we show how the firm with optimism and pessimism makes hedging decision under price ambiguity, thereby re-examining the validity of the full-hedging theorem. When the degree of ambiguity is higher, the firm is less confidence about its subjective estimation of the true output price distribution and thus is more likely to fully hedge the price risk. When the firm’s optimism level is higher, the firm assigns more weight to the best scenario and hence is more likely to choose a over-hedge or a under-hedge based on its optimism. Most notably, we identify a threshold of the firm’s optimism level above which the firm never chooses a full hedge, i.e., the full-hedging zone vanishes, even when there exist unbiased forward contracts (in the sense that \( E_g(\tilde{p}) = f_0 \)). This result is in sharp contrast with the extent literature, which shows that there always exists an interval or a point where full-hedging is optimal for the firm, i.e., a full-hedging zone always exists, when there exist unbiased hedging derivatives. Our results thus provide a novel explanation for why firms may shy away from full-hedging, i.e., the full-hedging theorem may fail based on the firm’s optimism and pessimism under price ambiguity.

\footnote{Under the Choquet expected utility theory pioneered by Schmeidler (1989), decision-makers’ beliefs about the likelihood of uncertain events are represented with a non-additive probability (capacity). Decision-makers are assumed to maximize the expected value of a utility function with respect to such capacity (the Choquet integral). The non-additivity of the capacity allows for different attitudes towards ambiguity. For example, a concave capacity reflects optimism while a convex capacity reflects pessimism. A neo-additive capacity is a weighting function, a simple variant of the well-celebrated inverse-S-shaped weighting function from the cumulative prospect theory.}

\footnote{This study thus complements Lien (2000) and Lien and Wang (2003), which verify the validity of the separation theorem within the maxmin expected utility model of Gilboa and Schmeidler (1989), and Iwaki and Osaki (2012) and Wong (2015), which show the robustness of the separation theorem within the smooth ambiguity aversion framework of Klibanoff et al. (2005).}
More generally, this paper is related to the literature of financial decision-making under ambiguity. While prior studies extensively examine investors’ behavior under ambiguity (e.g., Dow & Werlang, 1992; Cao et al., 2005; Easley & O’Hara, 2009; Bossaerts et al., 2010; Oh, 2014), this paper adds to the understanding of firms’ behavior under ambiguity by considering firms’ optimistic and pessimistic attitudes towards price ambiguity. Our study is also related to the literature of optimism in corporations. Economic theory (e.g., Brunnermeier & Parker, 2005), empirical research (e.g., Puri & Robinson, 2007), and psychology studies (e.g., Scheier et al. 2001) suggest that optimism and pessimism play a key role in decision-making. At corporate levels, we argue that, motivated by statistical evidence and market conditions, a firm may be able to estimate the uncertain output price distribution to some extent, but at the same time, the firm may suspect that its subjective estimation could turn out to be incorrect and hence optimistically (pessimistically) assign greater weight to more favorable (unfavorable) output price distributions. As such, optimism and pessimism can potentially influence a firm’s production and hedging decisions. While the existing literature focuses on how the optimism of the decision-maker influences a firm’s investment decisions (e.g., Heaton, 2002), our study sheds light how the optimism held by the decision-maker under price ambiguity affects a firm’s production and hedging decisions.

The rest of this paper is organized as follows. Section 2 presents the model. Section 3 analyzes the optimal production and hedging decisions for the firm. Section 4 provides numerical illustrations. Finally, Section 5 concludes. Omitted proofs are collected in the Appendix.

2 THE MODEL

We consider the competitive firm model of Sandmo (1970) within the context of the Choquet expected utility with neo-additive capacities (Chateauneuf et al., 2007). There are two dates, indexed by $t = 0, 1$. The firm produces a single commodity and incurs a deterministic cost, $c(q)$, where $q \geq 0$ is the output level. It is assumed that $c(q)$ satisfies the condition that $c(0) = c'(0) = 0$, $c'(q) > 0$, and $c''(q) > 0$ for all $q > 0$. The sale of the commodity occurs at time 1 at the then prevailing spot price, $\tilde{p}$, which is ex ante unknown but has a known support $[p_L, p_H]$, where $0 < p_L < p_H$. The discount rate is normalized to be zero and all payoffs and

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8See Epstein and Schneider (2010) and Etner et al. (2012) for literature reviews.

9Conceptually, the optimism in the current paper and these prior studies are in the same vein, i.e., the decision-maker assumes favorable outcomes while making decisions under uncertainty, although the rigorous definitions are not exactly the same.
costs are compounded to time 1. All results are qualitatively similar for positive discount rates.

To hedge against its exposure to the price risk, the firm can trade infinitely divisible commodity forwards at time 0, which provides a given price, $f_0$, for each unit of the commodity delivery at time 1. For ease of exposition, we restrict our attention to forward hedging.\footnote{This is as in Wong (2015).} Denote by $x$ the number of units of the commodity sold (purchased if negative) forward by the firm at time 0. The firm’s profit at time 1 is thus given by:

$$\tilde{\pi} = \tilde{p}(q - x) + f_0 x - c(q).$$

The price risk is ambiguous in the sense that the true probability distribution governing $\tilde{p}$ cannot be precisely specified and the firm thus holds multiple subjective probabilistic beliefs about $\tilde{p}$. Moreover, the firm holds optimistic and pessimistic attitudes towards price ambiguity. To capture these features, we model the decision-making of the firm à la the Choquet expected utility with neo-additive capacities theory (Chateauneuf et al., 2007):

$$\max_{x,q} V(x,q) = \theta E_g[u(\tilde{\pi})] + (1 - \theta) \left\{ \gamma \max_{m \in \Omega} E_m[u(\tilde{\pi})] + (1 - \gamma) \min_{m \in \Omega} E_m[u(\tilde{\pi})] \right\},$$

where $u$ is the Bernoulli utility function satisfying the condition that $u' > 0$ and $u'' < 0$, $g$ corresponds to the firm’s subjective estimation of the true probability distribution function governing $\tilde{p}$, $\theta \in [0,1]$ reflects the firm’s degree of confidence in this estimation, $(1 - \theta)$ measures the degree of ambiguity, which represents the degree to which the firm lacks confidence in $g$, $m$ represents an arbitrary probability distribution over the support $[p_L,p_H]$, $\Omega$ is the collection of all possible output price distributions, $\gamma \in [0,1]$ $(1 - \gamma)$ is the weight the firm assigns to the best (worst) scenario, reflecting the firm’s optimism (pessimism) under price ambiguity.\footnote{Note that one may interpret $(1 - \theta)$ as the degree of subjective ambiguity and $(p_H - p_L)$ as the degree of objective ambiguity. Nevertheless, as the focus of this study is on the role of $\gamma$ when $0 < \theta < 1$, we call $(1 - \theta)$ the degree of ambiguity and $(p_H - p_L)$ the range of the spot price.}

Thus, the firm chooses an output level $q$ and a hedging position $x$ to maximize a weighted sum of the expected utility, the maximum utility, and the minimum utility. Forward contracts are unbiased in the sense that $E_g(\tilde{p}) = f_0$.

This set-up is more general than prior studies in that Equation (2) encompasses some most popular models as special cases. If $\theta = 1$, i.e., there is no ambiguity, the model reduces to the standard expected utility model. If $\theta = 0$, i.e., the firm has no confidence about its subjective
estimation $g$, the firm follows the Arrow-Hurwicz criterion (Arrow & Hurwicz, 1972). If $\theta = 0$ and $\gamma = 0$, the model becomes the maxmin expected utility model (Gilboa & Schmeidler, 1989). When $0 < \theta < 1$, pure optimism holds if $\gamma = 1$ and pure pessimism prevails if $\gamma = 0$.

Our modelling approach allows us to explore the roles of optimism and pessimism under price ambiguity. According to Equation (2), the firm subjectively estimates the true probability distribution function governing $\tilde{p}$ to be $g$. The firm, however, is aware of its lack of relevant information. Thus, it only has a confidence level of $\theta$ about $g$ and gives weight of $(1 - \theta)$ to other possible output price distributions. The firm reacts to price ambiguity by further weighting the best and the worst scenarios among all possibilities. The relative weight on the favorable and the unfavorable outcomes depends on the parameter $\gamma$. The higher the level of $\gamma (1 - \gamma)$, the more weight is assigned by the firm to the favorable (unfavorable) outcome. Thus, following Chateauneuf et al. (2007), $\gamma (1 - \gamma)$ is interpreted as the firm’s level of optimism (pessimism) under price ambiguity.\footnote{Note that the weights, $\theta, 1 - \theta, \gamma,$ and $1 - \gamma,$ are not the subjective probabilities corresponding to the firm’s beliefs about the spot price in the future but rather the decision weights generated by a neo-additive capacity that represents its non-additive beliefs. This is a key difference between the Choquet expected utility with neo-additive capacities and the Savage subjective expected utility.}

Also, to focus on our main concerns, we restrict our attention to the case where $0 < \theta < 1$ in the following analysis.

3 THE ANALYSIS

Note that the best and worst scenarios for the firm are contingent on whether the firm chooses an under-hedge or an over-hedge. Hence, in the analysis, we first consider the case where $q > x$ and the case where $q < x$ separately. Then, we determine whether the firm will choose an under-hedge, an over-hedge, or a full-hedge (which applies when the necessary conditions for under-hedging and over-hedging both are invalid).

3.1 Under-Hedging

Consider first the case where $q > x$. Herein, the best outcome occurs at a degenerate probability that $m(p_H) = 1$ and the worst outcome occurs at a degenerate probability that $m(p_L) = 1$. The objective function of the firm (Equation 2) becomes:

$$V^u(x,q) = \theta E_g [u(\tilde{p}(q - x) + f_0x - c(q))]$$

$$+ (1 - \theta) \{\gamma u(p_H(q - x) + f_0x - c(q)) + (1 - \gamma)u(p_L(q - x) + f_0x - c(q))\}. \quad (3)$$
The first-order conditions for the optimal output level and the optimal forward position are given by:\(^{13}\)

\[
\theta E_g \left[ u' \left( \tilde{p} - c(q) \right) \right] + (1 - \theta) \left\{ \gamma u'_H \left( p_H - c(q) \right) + (1 - \gamma) u'_L \left( p_L - c(q) \right) \right\} = 0, \tag{4}
\]

\[
\theta E_g \left[ u' \left( f_0 - \tilde{p} \right) \right] + (1 - \theta) \left\{ \gamma u'_H \left( f_0 - p_H \right) + (1 - \gamma) u'_L \left( f_0 - p_L \right) \right\} = 0, \tag{5}
\]

where \( u' \equiv u' \left( \tilde{p}(q - x) + f_0x - c(q) \right), \ u'_H \equiv u' \left( p_H(q - x) + f_0x - c(q) \right), \) and \( u'_L \equiv u' \left( p_L(q - x) + f_0x - c(q) \right). \)

Summing up (4) and (5) yields:

\[
(f_0 - c(q)) \left\{ \theta E_g[u'] + (1 - \theta) \left( \gamma u'_H + (1 - \gamma) u'_L \right) \right\} = 0. \tag{6}
\]

Equation (6) implies that the optimal output level \( q^* \) solves \( c'(q^*) = f_0. \) Thus, the firm’s optimal output level depends neither on the output price distribution nor on the firm’s preferences (including the risk attitude, the degree of ambiguity, and the optimism and pessimism levels of the firm under price ambiguity). This result thus extends the separation theorem to the situation were the firm has the Choquet expected utility with neo-additive capacities under price ambiguity.

**Proposition 1** Consider a competitive firm that possesses the Choquet expected utility with neo-additive capacities. If the firm chooses to under-hedge, then the optimal output level is not affected by the firm’s preferences (including the risk attitude, the degree of ambiguity, and the optimism and pessimism levels of the firm under price ambiguity).

Now, we check whether \( q^* \) is indeed greater than \( x^* \) at optimum. Due to the concavity of \( V^u, \) it follows that:

\[
\text{sign} \left\{ x^* - q^* \right\} = \text{sign} \left\{ \frac{\partial V^u}{\partial x} \right\}_{x=q}. \tag{7}
\]

\(^{13}\)The second-order condition is ensured by the concavity of the objective function. In general, neo-additive capacities (and hence the Choquet integral) are neither convex nor concave (Chateauneuf et al., 2007). In our model, however, the objective functions for the firm, \( V^u \) (Equation (3)) and \( V^o \) (Equation (10)), are indeed concave. Note first that \( V^u \) and \( V^o \) are derived after solving the internal minimization problem. Thus, capacities are given when the firm makes production and hedging decisions (when we need to check the concavity of the objective function). Moreover, it is clear from (3) and (10) that \( V^u \) and \( V^o \) are linear combinations of three terms: \( u(\tilde{p}(q - x) + f_0x - c(q)), \ u(p_H(q - x) + f_0x - c(q)), \) and \( u(p_L(q - x) + f_0x - c(q)), \) where the coefficients are the associated decision weights assigned to different outcomes by the firm. Since these three terms are concave functions for given decision weights, \( V^u \) and \( V^o \) are ensured to be concave functions at the production and hedging stage.
Evaluating Equation (5) at $x = q$ yields:

$$
\left. \frac{\partial V^u}{\partial x} \right|_{x=q} = \theta E_g [f_0 - \tilde{p}] u'(f_0 q - c(q)) + (1 - \theta) \left\{ \gamma u'(f_0 q - c(q)) (f_0 - p_H) + (1 - \gamma) u'(f_0 q - c(q)) (f_0 - p_L) \right\}
$$

$$
= u'(f_0 q - c(q)) \left\{ [\theta + (1 - \theta) (\gamma + (1 - \gamma))] f_0 - [\theta E_g (\tilde{p}) + (1 - \theta) (\gamma p_H + (1 - \gamma) p_L)] \right\}.
$$

(8)

Because $u'(f_0 q - c(q)) > 0$ and $\theta + (1 - \theta) (\gamma + (1 - \gamma)) = 1$, it follows that:

$$
\text{sign} \left\{ \left. \frac{\partial V^u}{\partial x} \right|_{x=q} \right\} = \text{sign} \left\{ f_0 - [\theta E_g (\tilde{p}) + (1 - \theta) (\gamma p_H + (1 - \gamma) p_L)] \right\}.
$$

(9)

By Equations (7) and (9), we obtain the next proposition:

**Proposition 2** Under-hedging ($x^* < q^*$) is an optimum for $f_0 < \tilde{p}_u \equiv \theta E_g (\tilde{p}) + (1 - \theta) (\gamma p_H + (1 - \gamma) p_L)$.

Note that $\frac{\partial \tilde{p}_u}{\partial \gamma} > 0$, i.e., the threshold for under-hedging, $\tilde{p}_u$, increases with the level of the firm’s optimism, $\gamma$. It means that a more optimistic firm has a stronger tendency to under-hedge. Also, $\tilde{p}_u$ decreases (increases) with the degree of the firm’s ambiguity, $1 - \theta$, if $E_g (\tilde{p}) > (\gamma p_H + (1 - \gamma) p_L)$.\textsuperscript{14} It means that a firm facing greater ambiguity has a weaker (stronger) tendency to under-hedge if $E_g (\tilde{p}) > (\gamma p_H + (1 - \gamma) p_L)$. Moreover, $\frac{\partial^2 \tilde{p}_u}{\partial \gamma \partial \theta} < 0$. Together with the fact that $\frac{\partial \tilde{p}_u}{\partial \gamma} > 0$, it suggests that the under-hedging threshold is less sensitive to the firm’s optimism when the firm is more confident about its subjective estimation of the true probability distribution.

**Proposition 3** Suppose that under-hedging is optimal. Then, (i) a more optimistic firm has a stronger tendency to under-hedge; (ii) a firm facing greater ambiguity has a weaker (stronger) tendency to under-hedge if $E_g (\tilde{p}) > (\gamma p_H + (1 - \gamma) p_L)$; and (iii) the under-hedging threshold is less sensitive to the firm’s optimism when the firm is more confident about its subjective estimation of the true probability distribution.

### 3.2 Over-Hedging

Now consider the case where $q < x$. Herein, the best outcome occurs at a degenerate probability that $m(p_L) = 1$ and the worst outcome occurs at a degenerate probability that

\textsuperscript{14}Recall that $p_L < E_g (\tilde{p}) < p_H$. 

9
The objective function of the firm (Equation 2) becomes:

\[
V^o(x, q) = \theta E_g \left[ u(\tilde{p}(q - x) + f_0 x - c(q)) \right] \\
+ (1 - \theta) \left\{ \gamma u(p_L(q - x) + f_0 x - c(q)) + (1 - \gamma) u(p_H(q - x) + f_0 x - c(q)) \right\}. \tag{10}
\]

The analysis is similar to the previous section. First, the first-order conditions for the optimal output level and the optimal forward position yield \(c'(q^*) = f_0\).

**Proposition 4** Consider a competitive firm that possesses the Choquet expected utility with neo-additive capacities. If the firm chooses to over-hedge, then the optimal output level is not affected by the firm’s preferences (including the risk attitude, the degree of ambiguity, and the optimism and pessimism levels of the firm under price ambiguity).

It is evident that the separation theorem also holds for the full-hedging case. Together with Propositions 1 and 3, we hence show that the separation theorem is robust under price ambiguity within the context of the Choquet expected utility with neo-additive capacities. This complements Lien (2000) and Lien and Wang (2003), which verify the validity of the separation theorem within the maxmin expected utility model of Gilboa and Schmeidler (1989), and Iwaki and Osaki (2012) and Wong (2015), which show the robustness of the separation theorem within the smooth ambiguity aversion framework of Klibanoff et al. (2005).

Next, note that:

\[
\text{sign} \left\{ \frac{\partial V^o}{\partial x} \bigg|_{x=q} \right\} = \text{sign} \left\{ f_0 - [\theta E_g(\tilde{p}) + (1 - \theta)(\gamma p_L + (1 - \gamma)p_H)] \right\}. \tag{11}
\]

Due to the concavity of \(V^o\), we obtain the following proposition:

**Proposition 5** Over-hedging \((x^* > q^*)\) is an optimum for \(f_0 > \hat{p}_o \equiv \theta E_g(\tilde{p}) + (1 - \theta)(\gamma p_L + (1 - \gamma)p_H)\).

Note that \(\partial \hat{p}_o / \partial \gamma < 0\), i.e., the threshold for over-hedging, \(\hat{p}_o\), decreases with the level of the firm’s optimism, \(\gamma\). It means that a more optimistic firm has a stronger tendency to over-hedge.\(^{15}\) Also, \(\hat{p}_o\) decreases (increases) with the degree of the firm’s ambiguity, \(1 - \theta\), if \(E_g(\tilde{p}) > (<)\gamma p_L + (1 - \gamma)p_H\). It means that a firm facing greater ambiguity has a weaker

\(^{15}\)Note that, unlike in the under-hedging case, a higher level of optimism in the over-hedging case means that the firm assigns more weight to the event that \(\tilde{p} = p_L\). Note also the difference in the directions of the inequalities in Propositions 2 and 5.
(stronger) tendency to over-hedge if \( E_g(\bar{\rho}) > \gamma p_L + (1 - \gamma)p_H \). Moreover, \( \partial^2 \hat{p}_o / \partial \gamma \partial \theta > 0 \). Together with the fact that \( \partial \hat{p}_o / \partial \gamma < 0 \), it suggests that the over-hedging threshold is less sensitive to the firm’s optimism when the firm is more confident about its subjective estimation of the true probability distribution.

**Proposition 6** Suppose that over-hedging is optimal. Then, (i) a more optimistic firm has a stronger tendency to over-hedge; (ii) a firm facing greater ambiguity has a weaker (stronger) tendency to over-hedge if \( E_g(\bar{\rho}) > \gamma p_L + (1 - \gamma)p_H \); and (iii) the over-hedging threshold is less sensitive to the firm’s optimism when the firm is more confident about its subjective estimation of the true probability distribution.

### 3.3 Optimal Hedging Decision

We are now ready to determine whether the firm would choose to under-hedge, over-hedge, or full-hedge (which applies when the necessary conditions for under-hedging and over-hedging both are invalid).

First, simple algebra yields:

\[
\hat{p}_u - \hat{p}_o = (1 - \theta) \left[ \gamma (p_H - p_L) + (1 - \gamma) (p_L - p_H) \right]
= (1 - \theta) (p_H - p_L) (2 \gamma - 1)
> 0 \quad \text{iff} \quad \gamma > \frac{1}{2}.
\]

Next, by Propositions 2 and 5 and Equation (12), the firm’s optimal hedging decision can be analyzed based on three different scenarios: (a) \( 0 \leq \gamma < \frac{1}{2} \); (b) \( \gamma = \frac{1}{2} \); and (c) \( \frac{1}{2} < \gamma \leq 1 \), as in Figure 1. In what follows, we discuss these cases in detail.

#### 3.4 Case (a)

Case (a) considers \( 0 \leq \gamma < \frac{1}{2} \).\(^{16}\) It follows from Equation (12) that \( \hat{p}_u < \hat{p}_o \) in this case. According to Propositions 2 and 5, the firm chooses to under-hedge if \( f_0 < \hat{p}_u \) and over-hedge if \( f_0 > \hat{p}_o \). For \( \hat{p}_u \leq f_0 \leq \hat{p}_o \), the firm would full-hedge. Thus, the interval, \([\hat{p}_u, \hat{p}_o]\) represents the full-hedging zone in Case (a). Comparative statics show that the length of this full-hedging zone increases with \( \theta \) but decreases with \( \gamma \).\(^{17}\) It means that when \( \gamma < \frac{1}{2} \), the full-hedging

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\(^{16}\)This encompasses the special case of \( \gamma = 0 \), which is analyzed by Lien (2000).

\(^{17}\)Recall that the degree of ambiguity is measured by \( (1 - \theta) \).
zone shrinks with a higher level of optimism and expands with a higher degree of ambiguity.

3.5 Case (b)

Case (b) considers $\gamma = 1/2$. By Equation (12), we know that $\hat{p}_u = \hat{p}_o$ in this case. The firm chooses to full-hedge only if the forward price occurs at that particular point. That is, the full-hedging zone in Case (b) is one particular point. This is consistent with Iwaki and Osaki (2012) and Wong (2015), where they show that full-hedging occurs only when forward contracts are unbiased. For other forward prices, the firm chooses to under-hedge if $f_0 < \hat{p}_u = \hat{p}_o$ and over-hedge if $f_0 > \hat{p}_u = \hat{p}_o$.

3.6 Case (c)

The most notable case is Case (c), where $1/2 < \gamma \leq 1$. It follows from Equation (12) that $\hat{p}_o < \hat{p}_u$ in this case. By Propositions 2 and 5, the firm chooses to under-hedge if $f_0 < \hat{p}_u$ and over-hedge if $f_0 > \hat{p}_o$. Will the firm, however, optimally choose a full hedge for $f_0 \in [\hat{p}_o, \hat{p}_u]$? The answer is no.

For full-hedging to be optimal, it requires two conditions:

$$\left. \frac{\partial V_u}{\partial x} \right|_{x=q} \geq 0 \quad \text{and} \quad \left. \frac{\partial V^o}{\partial x} \right|_{x=q} \leq 0.$$  

However, taking the partial derivatives of Equations (3) and (10) with respect to $x$ and evaluating them at $x = q$ yield:

$$\left. \frac{\partial V_u}{\partial x} \right|_{x=q} < \left. \frac{\partial V^o}{\partial x} \right|_{x=q} \quad \text{if} \quad \gamma > \frac{1}{2}.$$  

It says that when the firm’s optimism level is greater than one-half, the two optimality conditions for full-hedging cannot not be satisfied simultaneously. Thus, full-hedging is never an optimal choice for the firm and is always dominated by either under-hedging or over-hedging. That is, there exists no full-hedging zone in Case (c).\(^{18}\)

To summarize, we present our key results in the following proposition and in Figure 1.

**Proposition 7** (i) For $0 \leq \gamma < 1/2$, the full-hedging zone expands with a higher level of ambiguity and shrinks with a higher level of optimism; (ii) for $\gamma = 1/2$, the full-hedging zone

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\(^{18}\)Our key point here is to show that there exists no full-hedging zone in Case (c). Appendix A contains a formal analysis of the firm’s behavior in Case (c) for $f_0 \in [\hat{p}_o, \hat{p}_u]$ and Section 4 provides illustrations.
degenerates to one particular point; and (iii) for $1/2 < \gamma \leq 1$, the full-hedging zone vanishes.

Intuitively, when the degree of ambiguity is higher, the firm is less confident about its subjective estimation of the true probability distribution governing output prices. Thus, it is more likely for the firm to choose a full-hedge to hedge the price risk. Also, when the firm’s optimism level is higher, the firm assigns more weight to the best scenario when it is uncertain about the true output price distribution. Hence, the firm is more likely to choose an over-hedge or an under-hedge, depending on whether it assigns more weight to $\hat{p} = p_L$ or $\hat{p} = p_H$. There, however, exists a threshold of the firm’s optimism level ($1/2$). When the firm’s optimism level is higher than the threshold, the firm assigns greater weight to the best scenario than to the worst scenario in the face of price ambiguity. Full-hedging becomes dominated by either over-hedging or under-hedging, depending on whether the firm assigns more weight to $\hat{p} = p_L$ or $\hat{p} = p_H$.

The key insight here is that there exists a threshold of the firm’s optimism level above which the full-hedging zone vanishes, i.e., firm never chooses to full-hedge even when there exist unbiased forward contracts (in the sense that $E_g(\tilde{p}) = f_0$). This result is in sharp contrast with prior studies (e.g., Lien, 2000; Lien & Wang, 2003; Isaki & Osaki, 2012; Wong, 2015). These prior studies all show that there always exists a full-hedging zone (an interval or a particular point where full-hedging is optimal) when hedging derivatives are unbiased. Our result thus identifies a novel explanation for why the full-hedging theorem may fail based on the roles of optimism and pessimism under price ambiguity.

4 NUMERICAL ILLUSTRATIONS

The central result of this paper is Proposition 7. This section demonstrates how it works by exercising simulations with exponential and quadratic utility functions. In what follows, the cost function is set to be $c(q) = 0.5q^2$, which implies that the optimal output level is $q^* = f_0$. For expositional simplicity, the uncertain price $\hat{p}$ takes on only two points, the lower bound ($p_L = 0.5$) and the higher bound ($p_H = 2.5$). Under the subjective estimation $g(.)$, each occurs with probability of 0.5. We then let $\theta$, $\gamma$, and $f_0$ vary. Finally, for both utility functions, we demonstrate how the level of the firm’s optimism affects its hedging choice.
4.1 Simulations: Exponential Utility

To bolster our analytical results, we provide some numerical simulations in Table 1, where \( u(\pi) = -exp(-\pi) \) and \( c(q) = q^2/2 \). We consider \( \theta = 0.2, 0.5, \) and 0.8. The column represents \( f_0 \) from 0.75 to 2.25 and the row represents \( \gamma \) from 0.1 to 0.9. For each row, the first line is \( x^* \) and the second line corresponds to \( V(x^*) \). We assume \( p_L = 0.5 \) and \( p_H = 2.5 \). The benchmark density function \( g(.) \) is \( Prob(\tilde{p} = p_H) = 1/2 \) and \( Prob(\tilde{p} = p_L) = 1/2 \).

To illustrate, we discuss Panel A when \( \theta = 0.2 \). When \( \gamma = 0.1 \), the optimal forward position is \( x^* = 0.5352 \) when \( f_0 = 0.75 \). Because our cost specification implies the optimal production level is \( q^* = f_0 \), the firm under-hedges. When \( f_0 = 1.0 \), \( x^* = 1 = q^* \) and there is a full-hedge. Similarly, the full-hedge solution applies when \( f_0 = 1.25, 1.50, 1.75, \) and 2.0. In the case that \( f_0 = 2.25, \ x^* = 2.4648, \) which is greater than \( q^* \). Thus, the firm over-hedges. Suppose now that \( \gamma \) increases to 0.3. The firm under-hedges when \( f_0 = 0.75 \) or 1.0, and over-hedges when \( f_0 = 2.0 \) or 2.25. The full-hedge solution holds if \( 1.25 \leq f_0 \leq 1.75 \). The full-hedging zone when \( \gamma = 0.3 \) is shorter than when \( \gamma = 0.1 \), showing that as \( \gamma \) increases, the range for full-hedging is
reduced.

When $\gamma = 0.5$, there is only one full-hedge solution (i.e., when $f_0 = 1.5$). Suppose that $\gamma = 0.7$. The firm under-hedges when $f_0 \leq 1.50$. At $f_0 = 1.5$, the firm achieves the same maximum expected Choquet utility at $x^* = 1.1683$ or $1.8316$. The former position is an under-hedge while the latter is an over-hedge. The firm can assume either position. Once $f_0$ exceeds 1.50, the firm will over-hedge. A similar case applies to $\gamma = 0.9$. Herein, the firm under (over) hedges when $f_0$ is smaller (larger) than 1.50. At $f_0 = 1.50$, the firm achieves the same maximum expected Choquet utility at $x^* = 0.7418$ or $2.2581$. Note that, as $\gamma$ increases, the firm takes more aggressive positions (i.e., under-hedges more or over-hedges more). The results for Panels B and C are similar to that for Panel A. Note that the full-hedging zone decreases as $\theta$ increases.

Furthermore, to check the robustness of our results, Table 2 considers a different estimated probability function $g(.)$ that $\text{Prob}(\tilde{p} = p_H) = 1/3$ and $\text{Prob}(\tilde{p} = p_L) = 2/3$ while maintaining the same assumptions about utility and cost functions as in Table 1. In Table 2, the triple represents $(f_0, x_u, x_o)$ and it shows that there is a different break point when a different $g(.)$ is used but the qualitative results remain. Thus, our results are robust to the change of the subjectively estimated probability function.

4.2 Simulations: Quadratic Utility

To further illustrate our points, we provide more numerical simulations in Table 3, where $u(\pi) = 25\pi - 0.5\pi^2$ and $c(q) = q^2/2$. We consider $\theta = 0.2, 0.5, \text{and} 0.8$. The column represents $f_0$ from 0.75 to 2.25 and the row represents $\gamma$ from 0.1 to 0.9. For each row, the first line is $x^*$ and the second line corresponds to $V(x^*)$. We assume $p_L = 0.5$ and $p_H = 2.5$. The benchmark density function $g(.)$ is $\text{Prob}(\tilde{p} = p_H) = 1/2$ and $\text{Prob}(\tilde{p} = p_L) = 1/2$.

First, we discuss Panel A with $\theta = 0.2$. When $\gamma = 0.1$, the optimal forward position is $x^* = -3.76$ if $f_0 = 0.75$. Our cost specification implies the optimal production level is $q^* = f_0$. Thus, the firm under-hedges. In fact, the firm buys instead of selling forwards. When $f_0 = 1.0$, $x^* = 1 = q^*$ and the firm engages in full-hedging. A similar full-hedge solution applies when $f_0 = 1.25, 1.50, 1.75, \text{and} 2.0$. In the case that $f_0 = 2.25$, $x^* = 6.35$, which is greater than $q^*$. Thus, the firm over-hedges. Suppose that $\gamma$ increases to 0.3. The firm under-hedges when
$f_0 = 0.75$ or $1.0$, and over-hedges when $f_0 = 2.0$ or $2.25$. The full-hedge solution holds if $1.25 \leq f_0 \leq 1.75$. The full-hedging zone when $\gamma = 0.3$ is shorter than when $\gamma = 0.1$, showing that as $\gamma$ increases, the range for full-hedging is reduced. When $\gamma = 0.5$, there is only one full-hedge solution (i.e., when $f_0 = 1.5$). Suppose that $\gamma = 0.7$. The firm under-hedges when $f_0 \leq 1.50$. At $f_0 = 1.5$, the firm achieves the same maximum expected Choquet utility at $x^* = -6.14$ or $9.14$. The former position is an under-hedge while the latter is an over-hedge. The firm can assume either position. Once $f_0$ exceeds 1.50, the firm will over-hedge. A similar case applies to $\gamma = 0.9$. Herein, the firm under-hedges (over-hedges) when $f_0$ is smaller (larger) than 1.50. At $f_0 = 1.50$, the firm achieves the same maximum expected Choquet utility at $x^* = -13.8$ or 16.8. Note that, as $\gamma$ increases, the firm takes more aggressive positions (i.e., under-hedges more or over-hedges more). The results for Panels B and C are similar to that for Panel A. Note that the full-hedging zone decreases as $\theta$ increases. For example, given that $\gamma = 0.1$, the full-hedging zone is $[1.0, 2.0]$ when $\theta = 0.2$ and $[1.25, 1.75]$ when $\theta = 0.5$. It reduces to a point, 1.50, when $\theta = 0.8$. Once again, these numerical illustrations give substance to our analytical results.

5 CONCLUSION

By embedding the Choquet expected utility with neo-additive capacities (Chateauneuf et al., 2007) into a competitive firm model (Sandmo, 1970), this paper analyzes the optimal production and hedging decisions of a competitive firm under price ambiguity when the firm is characterized by optimism and pessimism under price ambiguity. We show that the separation theorem remains intact, whereas the validity of the full-hedging theorem crucially depends on the level of the firm’s optimism under price ambiguity. Specifically, we show that, when the optimism level is less than one-half, full-hedging can prevail for a range of forward prices and the full-hedging zone expands with a higher level of ambiguity and shrinks with a higher level of optimism. When the optimism level equals one-half, the full-hedging zone degenerates to one particular point. Most notably, when the optimism level is greater than one-half, the full-hedging zone evaporates. While conventional wisdom holds that full-hedging is most reasonable when unbiased hedging derivatives markets exist, our results suggest that firms may deliberately choose not to full-hedge and it can potentially be attributed to their optimistic and pessimistic attitudes towards price ambiguity. This study thus identifies a novel circumstance under which the full-hedging
theorem fails and offers a new perspective on the roles of optimism and pessimism under price ambiguity in firms’ production and hedging decisions.

APPENDIX

A. The analysis of the firm’s hedging behavior when $1/2 < \gamma \leq 1$

We claim that there exists a threshold of forward prices, $\hat{f}_0 \in (\hat{p}_o, \hat{p}_u)$, such that the firm under-hedges if $f_0 < \hat{f}_0$ and over-hedges if $f_0 > \hat{f}_0$. When $f_0 = \hat{f}_0$, the firm may either under-hedge or over-hedge. The intuition for the existence of $\hat{f}_0$ is as follows. First, our analysis shows that, when $1/2 < \gamma \leq 1$, a small $\gamma$ pushes the firm to move closer to a full-hedge position while a larger $\gamma$ encourages the firm to take large over- or under-hedge positions. It is also shown that, when $1/2 < \gamma \leq 1$, full-hedging is always dominated by either under-hedging or over-hedging. Therefore, when $f_0$ moves along the real line (from the left to the right), there are only two remaining possibilities. Either the firm always under-hedges (or over-hedges) or the firm switches from under-hedging to over-hedging. The first case is not reasonable. Thus, we must have the second case.

Now we present the formal analysis. Since $c'(q^*) = f_0$, $q^* = (c')^{-1}(f_0)$. We can rewrite $V^u(x, q^*)$ as $V^u(x, f_0)$. Let $x^u(f_0) = \arg\max V^u(x, f_0)$. Similarly, we rewrite $V^\alpha(x, q^*)$ as $V^\alpha(x, f_0)$ and let $x^\alpha(f_0) = \arg\max V^\alpha(x, f_0)$. Recall that $\hat{p}_o < \hat{p}_u$ for $\gamma > 1/2$. Then, according to Equation (9), $x^u(\hat{p}_u) = q^*$ and $x^u(\hat{p}_o) < q^*$. Also, according to Equation (11), $x^\alpha(\hat{p}_o) = q^*$ and $x^\alpha(\hat{p}_u) > q^*$. These, together with Equations (3) and (10), imply that $V^u(x^u, \hat{p}_o) > V^\alpha(x^\alpha, \hat{p}_u)$ and $V^\alpha(x^\alpha, \hat{p}_o) > V^u(x^u, \hat{p}_u)$. We thus obtain that $V^u(x^u, f_0) - V^\alpha(x^\alpha, f_0)$ is positive when $f_0 = \hat{p}_o$ and negative when $f_0 = \hat{p}_u$. Because $V^u(x^u, f_0) - V^\alpha(x^\alpha, f_0)$ is a continuous function of $f_0$, there must be a point $\hat{f}_0 \in (\hat{p}_o, \hat{p}_u)$ such that $V^u(x^u, \hat{f}_0) = V^\alpha(x^\alpha, \hat{f}_0)$. Here, we assume the point $\hat{f}_0$ is unique, which is supported by both exponential and quadratic utility functions (see Appendix B and C). The conditions for the uniqueness depend upon the shape of the utility function and other parameters.\(^{19}\) Thus, the firm under-hedges if $f_0 < \hat{f}_0$ and over-hedges if $f_0 > \hat{f}_0$. When $f_0 = \hat{f}_0$, the firm may either under-hedge or over-hedge because the optimal under-hedging strategy provides the same expected utility as the optimal over-hedging strategy.

\(^{19}\)The technical issue is that while $V^\alpha$ is increasing in $f_0$, $V^u$ is increasing (decreasing) in $f_0$ if $x$ is positive (negative). Panel A of Table 4 contains both increasing and decreasing utility results. Also, the number of switch points must be odd. Suppose that $f_0$ is not unique. Then, there must be at least three of them between $p_o$ and $p_u$. The analysis becomes highly complicated.
strategy.\textsuperscript{20}

B. The proof that $\hat{f}_0$ is unique when utility is exponential

Consider the exponential utility function $u(\pi) = -\exp(-\pi)$\textsuperscript{21}. For the under-hedging case, let $\tilde{\pi}^u = (f_0 - \bar{p})x^u + \bar{p}q^* - c(q^*)$, $\pi^u_H = (f_0 - p_H)x^u + p_Hq^* - c(q^*)$, and $\pi^u_L = (f_0 - p_L)x^u + p_Lq^* - c(q^*)$. Then,

$$V^u(x^u, f_0) = \theta E_g \left[u(\tilde{\pi}^u)\right] + (1 - \theta) \left[\gamma u(\pi^u_H) + (1 - \gamma) u(\pi^u_L)\right].$$

(15)

By the envelope theorem, we obtain:

$$\frac{\partial V^u}{\partial f_0} = x^u \left\{ \theta E_g \left[u'(\tilde{\pi}^u)\right] + (1 - \theta) \gamma u'(\pi^u_H) + (1 - \theta)(1 - \gamma) u'(\pi^u_L) \right\},$$

(16)

which is positive because $u' > 0$.

Similarly, for the over-hedging case, we define $\tilde{\pi}^o$, $\pi^o_H$, and $\pi^o_L$ with $x^o$ replacing $x^u$. Thus,

$$V^o(x^o, f_0) = \theta E_g \left[u(\tilde{\pi}^o)\right] + (1 - \theta) \left[\gamma u(\pi^o_H) + (1 - \gamma) u(\pi^o_L)\right],$$

(17)

implying that:

$$\frac{\partial V^o}{\partial f_0} = x^o \left\{ \theta E_g \left[u'(\tilde{\pi}^o)\right] + (1 - \theta) \gamma u'(\pi^o_H) + (1 - \theta)(1 - \gamma) u'(\pi^o_L) \right\},$$

(18)

which is also positive because $u' > 0$.

Appendix A has shown that there exists a forward price, $\hat{f}_0$, such that $V^o(x^o, \hat{f}_0) = V^u(x^u, \hat{f}_0)$. Here we show that $\hat{f}_0$ is unique. To see this, note that, at $f_0 = \hat{f}_0$, $\frac{\partial V^u}{\partial f_0} = -x^u V^u(x^u, \hat{f}_0)$ and $\frac{\partial V^o}{\partial f_0} = -x^o V^o(x^o, \hat{f}_0)$. Consequently, at $f_0 = \hat{f}_0$,

$$\frac{\partial V^u}{\partial f_0} - \frac{\partial V^o}{\partial f_0} = -(x^u - x^o) V^o(x^o, \hat{f}_0),$$

(19)

\textsuperscript{20} Note that the firm changes its hedging behavior abruptly at $\hat{f}_0$. To see this, consider the case $\theta = 0$ and $\gamma = 1$. For such a firm, it trades in order to maximize the expected profit based on the optimism it holds. Given a $f_0$ less than $\hat{f}_0$, under-hedging to the maximum extent can be sustained as the optimum because the firm believes that the best scenario is $p = p_H$. Given a $f_0$ greater than $\hat{f}_0$, over-hedging to the maximum extent can be sustained as the optimum because the firm believes that the best scenario is $p = p_L$. At $f_0 = \hat{f}_0$, either under-hedging or over-hedging to the maximum extent can be sustained as the optimum. There is never a full-hedge. As such, the firm’s hedging decision and the associated weight assigned to the best scenario changes at $f_0 = \hat{f}_0$, which leads to the firm’s abrupt switch from under-hedging to over-hedging at $f_0 = \hat{f}_0$.

\textsuperscript{21} As is clear, the analysis can be easily extended to the case where $u(\pi) = -\exp(-\lambda \pi)$, where $\lambda > 0$ measures the degree of risk-aversion of the firm.
which is strictly negative because \( x^u < x^o \) and \( V^o(x^o, f_0) < 0 \) for the exponential utility function. It means that \( \hat{f}_0 \) is unique. Otherwise, there must exist another \( \tilde{f}_0 \) such that \( \frac{\partial V^u}{\partial f_0} - \frac{\partial V^o}{\partial f_0} > 0 \), which leads to a contradiction. We hence obtain that there exists a unique \( \hat{f}_0 \in (\hat{p}_o, \hat{p}_u) \) such that the firm under-hedges if \( f_0 < \hat{f}_0 \) and over-hedges if \( f_0 > \hat{f}_0 \). When \( f_0 = \hat{f}_0 \), the firm may either under-hedge or over-hedge and obtain the same expected utility.

C. The proof that \( \hat{f}_0 \) is unique when utility is quadratic

Consider the quadratic utility function \( u(\pi) = \pi - \frac{A}{2} \pi^2 \), where \( A > 0 \) measures the degree of risk-aversion of the firm. For the under-hedging case, let \( V^u(x) = \theta E_g [u(f_0 x + \hat{p}(q - x) - c(q))] + (1 - \theta) \gamma u(f_0 x + p_H(q - x) - c(q)) + (1 - \theta)(1 - \gamma) u(f_0 x + p_L(q - x) - c(q)) \). Denote the optimal hedging choice in this case by \( x^u \). It follows from the first-order condition that:

\[
x^u = q + \frac{(f_0 - \hat{p}_u) \left[ \frac{1}{A} - (f_0 q - c(q)) \right]}{\Delta_u},
\]

where \( \Delta_u = \theta E(f_0 - \hat{p})^2 + (1 - \theta) \gamma (f_0 - p_H)^2 + (1 - \theta)(1 - \gamma)(f_0 - p_L)^2 \).

Similarly, for the over-hedging case, we can obtain:

\[
x^o = q + \frac{(f_0 - \hat{p}_o) \left[ \frac{1}{A} - (f_0 q - c(q)) \right]}{\Delta_o},
\]

where \( \Delta_o = \theta E(f_0 - \hat{p})^2 + (1 - \theta) \gamma (f_0 - p_L)^2 + (1 - \theta)(1 - \gamma)(f_0 - p_H)^2 \).

Let \( M = f_0 q - c(q) \). We can write:

\[
f_0 x^u - c(q) = M + \frac{f_0 (f_0 - \hat{p}_u)(A^{-1} - M)}{\Delta_u}, \quad (22)
\]

\[
q - x^u = \frac{-(f_0 - \hat{p}_u)(A^{-1} - M)}{\Delta_u}, \quad (23)
\]

which implies that:

\[
f_0 x^u + \hat{p}(q - x) - c(q) = M + \frac{(f_0 - \hat{p})(f_0 - \hat{p}_u)(A^{-1} - M)}{\Delta_u}. \quad (24)
\]
Similarly, we have:

\[ f_0 x^u + p_H(q - x^u) - c(q) = M + \frac{(f_0 - p_H)(f_0 - \hat{p}_u)(A^{-1} - M)}{\Delta_u}, \quad (25) \]

\[ f_0 x^u + p_L(q - x^u) - c(q) = M + \frac{(f_0 - p_L)(f_0 - \hat{p}_u)(A^{-1} - M)}{\Delta_u}. \quad (26) \]

Substituting Equations (24), (25), and (26) into \( V^u \) yields:

\[ V^u(x^u) = M - \frac{AM^2}{2} - \frac{(f_0 - \hat{p}_u)^2(A^{-1} - M)(2 + A + AM)}{2\Delta_u}. \quad (27) \]

Similarly, we obtain:

\[ V^o(x^o) = M - \frac{AM^2}{2} - \frac{(f_0 - \hat{p}_o)^2(A^{-1} - M)(2 + A + AM)}{2\Delta_o}. \quad (28) \]

Equations (27) and (28) imply:

\[ V^u(x^u) = V^o(x^o) \iff \frac{(f_0 - \hat{p}_u)^2}{\Delta_u} = \frac{(f_0 - \hat{p}_o)^2}{\Delta_o}. \quad (29) \]

Now, by taking the partial derivative of \( V^u \) with respect to \( f_0 \) and by the definition of \( \hat{p}_u \), we obtain:

\[ \frac{\partial V^u}{\partial f_0} = \left[ 1 - A(f_0 x^u - c(q)) - A\hat{p}_u(q - x^u) \right] x^u. \quad (30) \]

By Equations (22) and (23), we can rewrite Equation (30) as:

\[ \frac{\partial V^u}{\partial f_0} = (1 - AM) \left[ 1 - \frac{(f_0 - \hat{p}_u)^2}{\Delta_u} \right] x^u. \quad (31) \]

Similarly, we obtain:

\[ \frac{\partial V^o}{\partial f_0} = (1 - AM) \left[ 1 - \frac{(f_0 - \hat{p}_o)^2}{\Delta_o} \right] x^o. \quad (32) \]

Taking the difference between Equations (31) and (32) yields:\(^{22}\)

\[ \frac{\partial V^u}{\partial f_0} - \frac{\partial V^o}{\partial f_0} = (1 - AM) \left[ 1 - \frac{(f_0 - \hat{p}_u)^2}{\Delta_u} \right] (x^u - x^o). \quad (33) \]

\(^{22}\)Note that \((1 - AM) \left[ 1 - \frac{(f_0 - \hat{p}_u)^2}{\Delta_u} \right] = (1 - AM) \left[ 1 - \frac{(f_0 - \hat{p}_o)^2}{\Delta_o} \right].\)
where \((1 - AM) > 0\) and \((x^u - x^o) < 0\). Appendix D shows that \(1 - \frac{(f_0 - \hat{p}_u)^2}{\Delta_u} > 0\). Thus, by Equation (33), \(\frac{\partial V^u}{\partial f_0} - \frac{\partial V^o}{\partial f_0} < 0\). As in the exponential utility case, this implies that there exists a unique \(\hat{f}_0 \in (\hat{p}_o, \hat{p}_u)\) such that the firm under-hedges for \(f_0 < \hat{f}_0\) and over-hedges if \(f_0 > \hat{f}_0\).

When \(f_0 = \hat{f}_0\), the firm may either under-hedge or over-hedge and obtain the same expected utility.

### D. The proof that \(1 - \frac{(f_0 - \hat{p}_u)^2}{\Delta_u} > 0\)

To determine the sign of \(1 - \frac{(f_0 - \hat{p}_u)^2}{\Delta_u}\), denote \(\mu_p = E_g[\tilde{p}]\) and recall that:

\[
f_0 - \hat{p}_u = \theta(f_0 - \mu_p) + (1 - \theta)\gamma(f_0 - p_H) + (1 - \theta)(1 - \gamma)(f_0 - p_L),
\]

\[
\Delta_u = \theta E[f_0 - \tilde{p}]^2 + (1 - \theta)\gamma(f_0 - p_H)^2 + (1 - \theta)(1 - \gamma)(f_0 - p_L)^2.
\]

Also,

\[
E[f_0 - \tilde{p}]^2 = E[(f_0 - \mu_p) + (\mu_p - \tilde{p})]^2 = (f_0 - \mu_p)^2 + E(\mu_p - \tilde{p})^2 > (f_0 - \mu_p)^2.
\]

Thus,

\[
\Delta_u > \theta(f_0 - \mu_p)^2 + (1 - \theta)\gamma(f_0 - p_H)^2 + (1 - \theta)(1 - \gamma)(f_0 - p_L)^2
\]

\[
\geq \theta(f_0 - \mu_p) + (1 - \theta)\gamma(f_0 - p_H) + (1 - \theta)(1 - \gamma)(f_0 - p_L)^2
\]

\[
= (f_0 - \mu_p)^2,
\]

where the second inequality follows from a mathematical property that \(\sum_{i=1}^{n} \alpha_i x_i^2 \geq (\sum_{i=1}^{n} \alpha_i x_i)^2\) for \(\sum_{i=1}^{n} \alpha_i = 1\). Equation (37) says that \(1 - \frac{(f_0 - \hat{p}_u)^2}{\Delta_u} > 0\).

### BIBLIOGRAPHY


Table 1: Exponential Utility (I)

### Panel A: $\theta = 0.2$

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<tr>
<th>$\gamma$</th>
<th>$f_0$</th>
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### Panel C: $\theta = 0.8$

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Note: In all panels, $u(\pi) = -\exp(-\pi)$, $q(\pi) = q^2/2$, $p_L = 0.5$, $p_H = 2.5$, $\Prob(\hat{\rho} = p_H) = 1/2$, and $\Prob(\hat{\rho} = p_L) = 1/2$. The column represents $f_0$ (the forward price at time 0) from 0.75 to 2.25 and the row represents $\gamma$ (the degree of optimism) from 0.1 to 0.9. For each row, the first line is $x^*$ (the optimal forward position) and the second line corresponds to $V(x^*)$ (the resulting Choquet expected value). For example, suppose $\theta = 0.2$. When $f_0 = 0.75$, $\gamma = 0.3$, the optimal forward position is 0.1087 and the corresponding Choquet expected utility is -0.6684 (note that the optimal output level is $q^* = f_0$). Similarly, suppose that $\theta = 0.8$. When $f_0 = 0.75$, $\gamma = 0.3$, the optimal forward position is -0.1428 and the corresponding Choquet expected utility is -0.5823.
Table 2: Exponential Utility (II)

<table>
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<th>$\gamma = 0.9$</th>
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<td>(1.6057, 1.3117, 1.8937)</td>
<td>(1.6295, 0.9670, 2.2571)</td>
</tr>
<tr>
<td>$\theta = 0.7$</td>
<td>(1.7357, 1.6068, 1.8622)</td>
<td>(1.7434, 1.4774, 1.9984)</td>
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</tbody>
</table>

Note: In this table, $u(\pi) = -\exp(-\pi)$, $c(q) = q^2/2$, $p_L = 0.5$, $p_H = 2.5$, $\text{Prob}(\tilde{p} = p_H) = 1/3$, and $\text{Prob}(\tilde{p} = p_L) = 2/3$. The triple represents $(f_0, x_u, x_o)$. 
## Table 3: Quadratic Utility

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<table>
<thead>
<tr>
<th>Panel B: $\theta = 0.5$</th>
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<td>$\gamma$</td>
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<table>
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<th>Panel C: $\theta = 0.8$</th>
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Note: In all panels, $u(\pi) = 25\pi - 0.5\pi^2$, $c(q) = q^2/2$, $p_L = 0.5$, $p_H = 2.5$, $\Prob(\tilde{p} = p_H) = 1/2$, and $\Prob(\tilde{p} = p_L) = 1/2$. The column represents $f_0$ (the forward price at time 0) from 0.75 to 2.25 and the row represents $\gamma$ (the degree of optimism) from 0.1 to 0.9. For each row, the first line is $x^*$ (the optimal forward position) and the second line corresponds to $V(x^*)$ (the resulting Choquet expected value). For example, suppose $\theta = 0.2$. When $f_0 = 0.75$, $\gamma = 0.3$, the optimal forward position is $-9.07$ and the corresponding Choquet expected utility is 59.18 (note that the optimal production is $q^* = f_0$). Similarly, suppose that $\theta = 0.8$. When $f_0 = 0.75$, $\gamma = 0.3$, the optimal forward position is $-10.73$ and the corresponding Choquet expected utility is 102.06.