Analysis of Optimal Debt Ratio in A Markov Regime-Switching Model

Wei-han Liu* Zhuo Jin†

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Abstract

A refined model for stochastic optimal control of debt management is proposed for optimal debt ratio estimation. This model considers productivity of capital, asset return, interest rate, and market regime switches. The decision maker aims to maximize the utility of terminal wealth by choosing the optimal debt ratio. A hidden Markov chain model and nonlinear filtering technique are included in this model for more accurate estimation of the actual situation. Using the dynamic programming approach, the value function obeys a Hamilton-Jacobi-Bellman equation (HJB). The empirical analyses on US markets are conducted and the observations are discussed.

Key Words. Stochastic optimal control, optimal debt ratio, hidden Markov chain model

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*Department of Finance, La Trobe University, Melbourne, Australia, weihanliu2002@yahoo.com.
†Department of Economics, The University of Melbourne, Melbourne, Australia, zjin@unimelb.edu.au.
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1 Introduction

The determination of optimal debt level has its short- and long-run effect on economic
development. More importantly, it is a crucial issue to national fiscal vulnerabilities, as
alerted by Greiner and Fincke (2009). Public concern escalates on the determination of
government debt especially when the major financial slumps hit. This issue is at large
connected with GDP, taxation, monetary policy, and government spending, i.e. affordability
and consumption of the public and the private sectors. Since the global financial crisis
in 2007, the international community has little choice but to print out more currency as a
feasible rescue measure. The additional capital pumped in to the markets is expected to play
a role in revitalizing the economy, in addition to its bailout function. However, issuing more
public debt does not necessarily deliver the expected policy effects, let alone the possible far-
reaching side effects. As the debt volume piles up, some undesirable outcomes are lurking,
e.g. inflation, maturity pressure of the debts, and incurred interest burden. The concern
mounts as the series of Qualitative Easing (QE) launch. There are at least two key issues in
the public debt management need to be addressed: the determination of the optimal debt
ratio and the optimal control of the debt. This study attempts to tackle those two issues
by presenting an innovative model of stochastic optimal control (SOC) and its empirical
analysis on the USA market. Accordingly we can make subjective evaluations of the QE
with respect to the optimal debt level.

The previous studies are mostly based on simplified assumptions or certain arbitrary the-
oretical frameworks (Barro (1979); Chari et al. (1991); Missale (1997); Aiyagari et al. (2022);
Schmitt-Grohé and Uribe (2004); Adam (2011). For example, government debt level opti-
mally follows a near random walk, or government spending evolves as an exogenous stochastic
process. The included variables follow certain deterministic models or interdependent paths.
The presented discussions focus more on the interaction among the variables with the optimal
The study of both issues is technically challenging. Among the representative piece of literatures, Stein (2012) have highlighted that there are some traditional assumptions cannot hold in this study. For example, each entity in the optimization process is far from isolation and significant interactions exist. Significant interaction among the units is present and ignorant of these feedbacks is at most to shoehorn the real world into one of the models to see how useful the approximation it is. Most of the variables in this study are far from being deterministic, e.g. series of price, interest rate, and net change in debt. Further, due to the complexity of the issues of optimal control of debt, dynamic programming techniques are essential to the possible solution. Alternatively, he proposes the SOC as a better technique simply because there are significant uncertainties in the variables and noticeable interactions among them. SOC is developed to solve the dynamic programming issue by choosing the control function that takes values in a metric space and influences of the state function which considers some process evolves over time with the dynamics given by a stochastic differential equation (SDE). Yet, there is significant room of refinement in the analysis framework by Stein (2012). For example, the author simplifies the study by formulating two models. Model I is a “Merton type” model which assumes constant productivity of capital and constant consumption ratio. That is an ergodic mean reverting process on price. Model I is based on the designed SDE for capital gain. Model II focuses on the two respective stochastic processes for capital gain and interest rate. The author works with both models that lead to a SDE formulation and allows for model uncertainty in deriving the optimal debt ratio.
However, the outcomes are based on the two models which are discussed separately. We believe that a holistic model can consider the interaction among the variables in a better manner in spite of the incurred complexity. It will thus demand a more advanced technique for this purpose.

Optimal control problems model defines a situation where a stochastic process is steered by the choice of a control function that takes values in a compact metric space, see Caputo (2005). The HJB is central to optimal control theory. When solved locally, the HJB is a necessary condition. When solved over the whole of state space, the HJB equation is a necessary and sufficient condition for an optimum. The solution is an open loop, but it also permits the solution of the closed loop problem. The HJB method can be generalized to stochastic systems as well, i.e. SOC. SOC can be applied to nonlinear systems with random inputs and either perfect or imperfect measurements. Though complex, a HJB based on expectations can be solved analytically. The explicit form of solution can be obtained with the boundary conditions. Consequently, the derived optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision (Bellman (2010)). This process refers to principle of optimality.

There are two principal methods to solve for the SOC problem: Pontryagins maximum principle (Pontryagin (1959)) and Bellmans dynamic programming (Bellman (2010)). Pontryagins maximum principle states that any optimal control along with the optimal state trajectory will solve the so-called Hamiltonian system which consists of a collection of adjoint equations and the maximum conditions. Bellmans dynamic programming breaks a dynamic optimization problem into simpler subproblems. Bellmans dynamic programming

\footnote{There are two types of forms for the optimal value of the control function: open-loop form, feedback or closed-loop form. Essentially, the feedback form of the optimal control is a decision rule, for it gives the optimal value of the control for any current period and any admissible state in the current period that may arise. In contrast, the open-loop form of the optimal control is a curve, for it gives the optimal values of the control as the independent variable time varies over the planning horizon. Even though both controls differ in form, they yield identical values for the optimal control at each date of the planning horizon.}
relies on the principle of optimality and establishes the relationships among a family of optimal control problems with different initial time and states via Hamilton-Jacobi-Bellman (HJB) equation. The two methods are equivalent in the sense that the Hamilton system can be deduced from the HJB equation, and vice versa. See Yong and Zhou (1999) for details. While both principle methods yield the same results, Bellmens dynamic programming exploits the recursive nature of the problem and characterizes the maximum value of the objective function, i.e. the so-called value function, as a function of the state of the system. The solutions by HJB give the optimal solutions to SOC. We thus resort to the Bellmans technique. Noteworthy, there are a number of significant and new developments in HJB equation. The recent applications cover some issues of dynamic programming in finance (e.g. Crandall and Lions (1983)) and actuarial science (e.g. Asmussen and Taksar (1997) and Jin et al. (2013)). We plan to apply the new developments of dynamic programming to the optimal control of debt.

While Stein (2012) assumes the ratio constant to simplify the SOC issue, we adopt a stochastic consumption ratio which is described with a diffusion process. Moreover, we extend the SOC to incorporate a hidden Markov chain representing the discrete movements. That is, the jumps describe the economic trends and impacts that cannot be modeled as either ordinary or stochastic differential equations. A typical example is a two-state Markov chain with one representing the bull market and the other representing the bear market. See also Hamilton (1989) and Zhou and Yin (2003). Then, the return rate of assets is modeled as a switching diffusion process modulated by a hidden two-state Markov chain in this paper. We expect that the two introduced refinements make our proposed model outperform in depicting the actual market condition and performing SOC. However, the state of the Markov chain is observable with additive white noise as specified in (4.3). In reality, an observation without noise is virtually impossible to detect. Since the state in the Markov chain are not directly observable, the nonlinear filtering technique is introduced.
to overcome this estimation difficulty so as to recover necessary information of observations (Liptser and Shiryaev (1968)). Among the latest developments, the well-known filter by Wonham (1965) is adopted in this study and accordingly the partially observed system will be converted to a completely observable one. Consequently the feasibility of the estimation of our proposed model is assured. Related work can be referred to Haussmann and Sass (2004) and Rishel and Helmes (2006). Further, we proceed to solve the HJB equations with given boundary conditions so as to solve the optimal control of the problem. We have derived the formulae for the time-varying optimal debt ratios of the bear market, the bull markets, and the aggregate market. The sensitivity analyses in terms of the respective variable are assured.

2 Formulation

The wealth process $X(t)$ is described as the difference between the asset value and debt, where the asset value and debt are denoted by $K(t)$ and $L(t)$, respectively. Let $P(t)$ be the price of the asset and $Q(t)$ be the quantity of the asset, and the asset value equals the product of asset price and quantity. That is,

$$X(t) = K(t) - L(t) = P(t)Q(t) - L(t).$$

(2.1)

We assume that the asset price $P(t)$ in the financial market follows the Geometric Brownian Motion process

$$\frac{dP(t)}{P(t)} = \mu(t)dt + \sigma_1 dW_1(t),$$

(2.2)

where $\mu(t)$ is the return rate of the asset and $\sigma_1$ is the corresponding volatility and $W_1(t)$ is a standard Brownian motion. Because of the randomness of the environment, the yield rate of the asset is not constant and can be effected by the market modes or other economic
factors. Hence, the net change of asset value follows

\[ dK(t) = P(t)dQ(t) + dP(t)Q(t) \]
\[ = P(t)dQ(t) + \frac{dP(t)}{P(t)}K(t) \]
\[ = P(t)dQ(t) + K(t)[\mu(t)dt + \sigma_1dW_1(t)], \tag{2.3} \]

Note that the net change of wealth in composed of two parts. \( K(t)[\mu(t)dt + \sigma_1dW_1(t)] \) comes from the price change \( dP(t) \) which leads to the capital gain or loss of the asset value. \( P(t)dQ(t) \) is the change due to the quantity change of the asset volume. It is considered as the investment and also included in the expenditure.

The net debt \( L(t) \) equals expenditure less income. The net change of expenditure in a short period \([t, t + dt]\) includes the interest of debt with at the rate \( r(t) \), the investment part \( P(t)dQ(t) \), and the consumption \( dC(t) \) such that \( dC(t) = c(t)X(t)dt \), where \( c(t) \) is the consumption ratio. Income is described as the product of productivity rate of capital and asset value. Let \( \beta(t) \) be the productivity rate of capital and \( Y(t) \) be the income. Then \( dY(t) = \beta(t)K(t)dt \). Now we obtain the net change of debt

\[ dL(t) = r(t)L(t)dt + P(t)dQ(t) + dC(t) - dY(t) \]
\[ = r(t)L(t)dt + P(t)dQ(t) + c(t)X(t)dt - \beta(t)K(t)dt \tag{2.4} \]

Interest rate of the debt is largely influenced by the randomness of economic environment and can be described as a random variable. In what follows, the interest rate \( r(t) \) is assumed to follow a diffusion process

\[ r(t)dt = rdt + \sigma_2(\rho_1dW_1(t) + \sqrt{1 - \rho_1^2}dW_2(t)), \tag{2.5} \]
where $r$ is the net expected interest rate of the debt, $\sigma_2$ represents the volatility of the interest rate, $W_2(t)$ is a standard Brownian motion which is independent with $W_1(t)$. $-1 < \rho_1 < 1$ represents instantaneous correlation between the growth rate of asset value and interest rate of debt. To proceed, we adopt the assumption that the consumption is paid from the current productivity of asset, see Stein (2012). However, unlike the model in Stein (2012), the consumption ratio cannot be deterministic and is correlated to the asset value and interest rate of debt. Hence, the consumption ratio $c(t)$ can be described as a function of the productivity ratio $g(\beta(t))$ and subject to a random perturbation. In Stein (2012), $g(\beta(t))$ equals $\beta(t)$ when $\beta(t) \geq 0$, and equals 0 when $\beta(t) < 0$. It shows that all of the positive productivity of asset are consumed, and no consumption is considered when productivity is negative. With a $g$ function defined, the relationship between productivity and consumption is more versatile and realistic in our model. A simple case is that a linear $g$ function which will represent the consumption equals proportion or multiple of the productivity. The random perturbation is described as a three-dimensional Brownian motion. It follows

$$
c(t)dt = g(\beta(t))dt + \sigma_3 \left( \rho_2 dW_1(t) + \frac{\rho_3 - \rho_1 \rho_2}{\sqrt{1 - \rho_1^2}} dW_2(t) + \left[ 1 - \rho_2^2 - \frac{(\rho_3 - \rho_1 \rho_2)^2}{\sqrt{1 - \rho_1^2}} \right] dW_3(t) \right),
$$

(2.6)

$W_3(t)$ is a standard Brownian motion which is independent with $W_1(t)$ and $W_2(t)$. The second term on the right of (2.6) is formulated to show the correlation between consumption with asset value and interest rate of debt. The volatility of consumption ratio is modeled by the sum of three independent Brownian motions. In view of (2.6), the volatility of consumption ratio equals $\sigma_3$. $\rho_2$ represents the correlation between consumption ratio and growth rate of asset value. $\rho_3$ represents the correlation between consumption ratio and interest rate of debt. In conclusion, we obtain the correlation between the random variables.
That is,

\[ \text{Cov} \left( \frac{dK(t)}{K(t)}, r(t)dt \right) = \rho_1 \sigma_1 \sigma_2 \sigma_3 dt, \]

\[ \text{Cov} \left( \frac{dK(t)}{K(t)}, c(t)dt \right) = \rho_2 \sigma_1 \sigma_3 \sigma_4 dt, \]

\[ \text{Cov}(r(t)dt, c(t)dt) = \rho_3 \sigma_2 \sigma_3 \sigma_4 dt. \] (2.7)

Combining (2.4)-(2.6), the net change of debt equals

\[ dL(t) \]

\[ = [r(t)L(t) + g(\beta(t))X(t) - \beta(t)K(t)]dt + P(t)dQ(t) + (L(t)\rho_1 \sigma_2 + X(t)\rho_2 \sigma_3)dW_1(t) + \left( L(t)\sigma_2 \sqrt{1 - \rho_1^2} + X(t)\sigma_3 \frac{\rho_3 - \rho_1 \rho_2}{\sqrt{1 - \rho_1^2}} \right) dW_2(t) + X(t)\left( 1 - \rho_2^2 - \frac{(\rho_3 - \rho_1 \rho_2)^2}{\sqrt{1 - \rho_1^2}} \right) dW_3(t). \] (2.8)

In view of (2.1), (2.3) and (2.8), the wealth process \( X(t) \) follows

\[ dX(t) \]

\[ = [K(t)(\mu(t) + \beta(t)) - r(t)L(t) - g(\beta(t))X(t)]dt + (K(t)\sigma_1 - L(t)\rho_1 \sigma_2 - X(t)\rho_2 \sigma_3)dW_1(t) - \left( L(t)\sigma_2 \sqrt{1 - \rho_1^2} + X(t)\sigma_3 \frac{\rho_3 - \rho_1 \rho_2}{\sqrt{1 - \rho_1^2}} \right) dW_2(t) - X(t)\left( 1 - \rho_2^2 - \frac{(\rho_3 - \rho_1 \rho_2)^2}{\sqrt{1 - \rho_1^2}} \right) dW_3(t). \] (2.9)
Let \( \pi(t) = \frac{L(t)}{X(t)} \) be the debt ratio, (2.9) can be written as

\[
\frac{dX(t)}{X(t)} = \left[ \pi(t)(\mu(t) + \beta(t) - r(t)) + \mu(t) + \beta(t) - g(\beta(t)) \right] dt + \left( \pi(t)(\sigma_1 - \rho_1 \sigma_2) + \sigma_1 - \rho_2 \sigma_3 \right) dW_1(t)
\]

\[
- \left( \pi(t) \sigma_2 \sqrt{1 - \rho_1^2} + \sigma_3 \frac{\rho_3 - \rho_1 \rho_2}{\sqrt{1 - \rho_1^2}} \right) dW_2(t) - \left( 1 - \rho_2^2 - \frac{(\rho_3 - \rho_1 \rho_2)^2}{\sqrt{1 - \rho_1^2}} \right) dW_3(t).
\]

We are now working on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\), where \( \mathcal{F}_t \) is the \( \sigma \)-algebra generated by \( \{W_1(s), W_2(s), W_3(s) : 0 \leq s \leq t\} \).

The decision maker will choose the optimal debt ratio to maximize the return to the shareholders. To measure the performance of management, performance function can be described as a utility function of terminal wealth. Denote by \( U(\cdot) \) the utility function. For an arbitrary admissible strategy \( \pi \), the performance function \( J(\cdot) \) follows

\[
J(x, \pi) = E_x[U(X(T))].
\]

Suppose that \( \mathcal{A} \) is the collection of all admissible strategies. Define the value function as

\[
V(x) := \sup_{\pi \in \mathcal{A}} J(x, \pi).
\]

10
For simplicity, we let

\[ b(t) = \pi(t)(\mu(t) + \beta(t) - r(t)) + \mu(t) + \beta(t) - g(\beta(t)) \]

\[ m_1 = \pi(t)(\sigma_1 - \rho_1 \sigma_2) + \sigma_1 - \rho_2 \sigma_3, \]

\[ m_2 = \pi(t)\sigma_2 \sqrt{1 - \rho_1^2} + \sigma_3 \frac{\rho_3 - \rho_1 \rho_2}{\sqrt{1 - \rho_1^2}}, \]

\[ m_3 = 1 - \rho_2^2 - \frac{(\rho_3 - \rho_1 \rho_2)^2}{\sqrt{1 - \rho_1^2}}. \]

\( b(t) \) can be considered as the drift of the net wealth, and \( m_1, m_2, m_3 \) represents the volatilities from each Brownian motion specified in (2.6). Then (2.10) can be simplified as

\[ \frac{dX(t)}{X(t)} = b(t)dt + m_1 dW_1(t) - m_2 dW_2(t) - m_3 dW_3(t). \]

Assume \( V(\cdot) \) is a sufficiently smooth function, define an operator \( \mathcal{L}^\pi \) such that

\[ \mathcal{L}^\pi V(x) = x b(t)V_x(x) - \frac{1}{2} (m_1^2 + m_2^2 + m_3^2)x^2 V_{xx}(x), \]

where \( V_x \) and \( V_{xx}(x) \) denote the first and second derivatives with respect to \( x \), respectively. By applying the dynamic programming principle, the value function (2.12) satisfies the HJB equation when

\[ \mathcal{L}^\pi V(x) = 0, \]

Suppose there is an admissible feedback control \( \pi^* (\cdot) \) that is the maximizer of (2.15). Then it can be shown that \( V(x) \) is indeed the optimal cost and \( \pi^*(t) \) is the optimal control. In fact, let \( \hat{\pi}(t) \), other than the optimal \( \pi^*(t) \), be an arbitrary admissible control whose
trajectory is \( \hat{x}(t) \). In view of (2.15),

\[
0 = \mathcal{L}^\pi V(x),
\]

\[
0 \geq \mathcal{L}^{\hat{x}} V(\hat{x}).
\]

for all values of \( x, \hat{x} \in (0, \infty) \) and \( t > 0 \). Applying Dynkin’s formula to \( V(x) \) specified in (2.12), we have

\[
E_x V(x(T)) - V(x) = E_x \int_0^T \mathcal{L}^\pi V(x(s)) ds = 0.
\]  

(2.16)

In view of (2.16), we obtain

\[
V(x) = E_x V(x(T)).
\]

Similarly to (2.16) in the case of \( \hat{x} \), we also have

\[
E_x V(\hat{x}(T)) - V(x) = E_x \int_0^T \mathcal{L}^{\hat{x}} V(\hat{x}(s)) ds \leq 0.
\]

Hence, we obtain

\[
V(x) \geq E_x V(\hat{x}(T)).
\]

Thus,

\[
J(x, \hat{\pi}) \leq V(x, i) = J(x, \pi^*).
\]

Hence the maximizing control \( \pi^*(\cdot) \) is optimal.

3 Growth of Net Wealth

The representative financial institution is risk averse and assumed to maximize the return of the shareholders or the wealth of the firm. There are some utility functions that
can be selected as criterion function in the optimization process. For those which can be solved analytically, the major selections include logarithmic, exponential, CARA (Constant Absolute Risk Aversion), and HARA (Hyperbolic Absolute Risk Aversion). Among them, Stein (2012) recommends the logarithmic function at least for the following advantages: (1) Severe penalty for significant loss in the logarithmic function; (2) The risk coefficients can be arbitrary in specifying levels of risk aversion; (3) SOC can be derived directly by integration of the Ito equation. We follow his choice.

To maximize the growth of net wealth, a natural choice is to maximize the expectation of the logarithm of net wealth over a finite period \( T \). Without loss of generality, we assume the net wealth at initial time \( X(0) = 1 \). Hence, the objective function (2.11) becomes

\[
J(x, \pi) = E_x[\ln(X(T))].
\]  

(3.1)

In this formulation, the goal is to maximize the expected growth of net worth. Comparing with (2.11), (3.1) coincides with a log-utility function in the formulation. Our aim is equivalent to maximize the expectation of the log-utility of terminal wealth. Log-utility function is a concave objective function with positive first order derivative and negative second order derivation. While the decision maker in our formulation wants the financial institution to generate a high return to the shareholders, extremely heavy penalty will be placed on a debt that would lead to bankruptcy (zero net worth), where \( \ln(0) = -\infty \). Moreover, even when the company is not strictly bankrupt but with a serious financial status, where \( x \) is close to zero, the objective function will give a very large negative number. Hence, the decision maker is very risk averse.
Applying Itô’s formula to (3.1), we obtain

\[
\ln(X(T)) = \ln(X(0)) + \int_0^T [\pi(t)(\mu(t) + \beta(t) - r(t)) + \mu(t) + \beta(t) - g(\beta(t)) - \frac{1}{2}(m_1^2 + m_2^2 + m_3^2)] \, dt + M_t,
\]

where

\[
M_t = \int_0^T m_1 \, dW_1 - \int_0^T m_2 \, dW_2 - \int_0^T m_3 \, dW_3.
\]

Note that \(M_t\) is a summation of three integrals of independent Brownian motions and qualified as a martingale. Hence, the HJB equation (2.15) follows

\[
\max_{\pi \in A} \pi(t)(\mu(t) + \beta(t) - r(t)) + \mu(t) + \beta(t) - g(\beta(t)) - \frac{1}{2}(m_1^2 + m_2^2 + m_3^2) = 0 \quad (3.3)
\]

That is, the maximizer \(\pi^*\) satisfies

\[
\frac{\partial}{\partial \pi} \left\{ \left( \mu(t) + \beta(t) - g(\beta(t)) \right) - \frac{1}{2} \left[ (\sigma_1 - \rho_2 \sigma_3)^2 + \frac{\sigma_3^2 (\rho_3 - \rho_1 \rho_2)^2}{1 - \rho_1^2} + \left( 1 - \rho_2 - \frac{(\rho_3 - \rho_1 \rho_2)^2}{\sqrt{1 - \rho_1^2}} \right)^2 \right] 
\right\}_{\pi = \pi^*} = 0 \quad (3.4)
\]

The optimal debt ratio follows

\[
\pi^* = \frac{\mu(t) + \beta(t) - r(t) - \left( \sigma_1 - \rho_1 \sigma_2 \right) \left( \sigma_1 - \rho_2 \sigma_3 \right) - \sigma_2 \sigma_3 (\rho_3 - \rho_1 \rho_2)}{\left( \sigma_1 - \rho_1 \sigma_2 \right)^2 + \sigma_2^2 (1 - \rho_1^2)}. \quad (3.5)
\]

The optimal debt ratio \((\pi^* \text{ or } \pi^*(t))\) is time-varying with respect to \(\mu, \beta, \text{ and } r\). However, for simplification, we will write \(\pi^*\) instead of \(\pi^*(t)\). We will omit \((t)\) for all optimal debts ratios afterwards.

\(J'(x, \pi)\) and \(J''(x, \pi)\) denote the first and second derivatives of \(J(x, \pi)\) to \(\pi\), respectively. In view of (3.4), we have \(J'(x, \pi^*) = 0\), and \(J''(x, \pi^*) = -[(\sigma_1 - \rho_1 \sigma_2)^2 + \sigma_2^2 (1 - \rho_1^2)]T\). On
the other hand, according to Taylor expansion, \( J(x, \pi) \) can be approximated as

\[
J(x, \pi) = J(x, \pi^*) + J'(x, \pi^*)(\pi - \pi^*) + \frac{1}{2}J''(x, \pi^*)(\pi - \pi^*)^2 + o((\pi - \pi^*)^2),
\]  

(3.6)

where \( o((\pi - \pi^*)^2) \) means something negligible as compared to \( (\pi - \pi^*)^2 \). Denote by \( \Psi(t) = \pi(t) - \pi^*(t) \) as the excess debt ratio, the difference between the actual ratio and its optimal level. Then the difference between maximal and actual expected growth of performance function (2.11) is a quadratic function of excess debt. That is,

\[
J(x, \pi^*) - J(x, \pi) = \frac{1}{2}\Psi^2T[(\sigma_1 - \rho_1\sigma_2)^2 + \sigma_2^2(1 - \rho_1^2)].
\]  

(3.7)

In view of (3.7), we see that the right side of is nonnegative. It means that the decision maker will always bear the potential loss if he cannot choose the optimal debt ratio \( \pi^* \). The loss will be quadratically enlarged if \( \pi \) diverge from \( \pi^* \). The loss will be minimized when the optimal debt ratio is achieved, and the net wealth will be maximized at the same time.

On the other hand, in view of (3.2), the variance of the expected growth rate of wealth \( \text{Var}(\ln(X(T))) \) is a function of \( \pi \) and follows

\[
\text{Var}(\ln(X(T))) = \text{Var}(M_t) = (m_1^2 + m_2^2 + m_3^2)T,
\]

where

\[
m_1^2 + m_2^2 + m_3^2 \geq 2[\pi^*(\mu(t) + \beta(t) - r(t)) + \mu(t) + \beta(t) - g(\beta(t))].
\]

The inequality equals when \( \pi = \pi^* \). It shows that the expected growth of wealth is more vulnerable to shocks if the actual debt ratio diverge from the optimal debt ratio from the above. Moreover, in reality, the optimal debt ratio is hardly achieved due to the observation perturbation. One can deduce the optimal debt ratio theoretically based on the perturbed data, but the error to the optimal always exists as \( \frac{1}{2}\Psi^2[(\sigma_1 - \rho_1\sigma_2)^2 + \sigma_2^2(1 - \rho_1^2)] \), which is
positive in unit time length.

4 Markov Jump Process

It is advisable to consider discrete movements for better modeling performance, such as random environment, market trends, business cycles, etc. One of the powerful techniques is to use Markov switching models, where continuous dynamics and discrete events coexist in the systems. We assume the return rate of asset $\mu(t)$ follows a two-state Markov jump process, which has the generator

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix},$$

where frequencies $\lambda_1 > 0$ and $\lambda_2 > 0$. The return series $\mu(t)$ takes values in the state space $\{k_1, k_2\}$, where $k_1 > 0$, and $k_2 < 0$. We use $k_1 > 0$ to represent a bull market and $k_2 < 0$ represents a bear market. In view of the generator of Markov chain, for small time interval $\delta > 0$, we have

$$\Pr\{\mu(t + \delta) = k_2 | \mu(t) = k_1, \mu(s), s \leq t\} = \begin{cases} 
\lambda_1 \delta + o(\delta), & \text{if } k_2 \neq k_1, \\
1 - \lambda_1 \delta + o(\delta), & \text{if } k_2 = k_1, 
\end{cases} \quad (4.1)$$

and

$$\Pr\{\mu(t + \delta) = k_1 | \mu(t) = k_2, \mu(s), s \leq t\} = \begin{cases} 
\lambda_2 \delta + o(\delta), & \text{if } k_1 \neq k_2, \\
1 - \lambda_2 \delta + o(\delta), & \text{if } k_1 = k_2. 
\end{cases} \quad (4.2)$$
It shows that the expected time to stay in the bull market is $1/\lambda_1$ and the expected time to stay in the bear market is $1/\lambda_2$, see Hamilton (1989) for details.

In practice, we do not have direct information of the Markov jump process, and can only observe it with its white noise innovation. That is, we observe $f(t)$, whose dynamics is given by

$$f(t) = \int_0^t [\mu(s)ds + \sigma_1 dW_1(t)].$$

(4.3)

To estimate the dynamic state of the Markov jump process on the basis of the data perturbed by white noise, we resort to the technique of nonlinear filtration, introduced by Wonham (1965). This technique reduces the partially observed problem to a completely observed problem.

The first key issue for the estimation of Markov process can be obtained if we have the conditional probability of the states of the Markov process based on the available data up to time. Let $q(t)$ be the conditional probability of state of the Markov jump process that represents the rate of return performs in a bull market. That is,

$$q(t) = \mathbb{P}[\mu(t) = k_1|\mathcal{F}_s, 0 \leq s \leq t],$$

(4.4)

with $q_0 = \mathbb{P}[\mu(0) = k_1]$. $q_0$ is the initial point of conditional probability and $q(t) \in [0, 1]$. In light of the nonlinear filtering results in Wonham (1965), the conditional probability $q(t)$ is a solution of

$$dq(t) = [-\lambda_1 q(t) + \lambda_2 (1 - q(t))]dt + \frac{k_1 - k_2}{\sigma_1}q(t)(1 - q(t))dB(t),$$

(4.5)

where $q(0) = q_0$, and $B(t)$ is a Brownian motion which is also called the innovations process, see also Davis and Markus (1981); Rishel (2006). The innovations process $B(t)$ is independent with $W_2(t)$ and $W_3(t)$. Hence, in view of the conditional probability defined in (4.4), (2.2)
can be rewritten as follows based on the conditional probability and innovation process. That is,

\[
\frac{dP(t)}{P(t)} = k_1 q(s) + k_2 (1 - q(s)) ds + \sigma_1 dB(t).
\]

(4.6)

By applying Itô’s formula to (3.1), we obtain

\[
\ln(X(T)) = \ln X(0) + \int_0^T \{ \pi(t) [k_1 q(t) + k_2 (1 - q(t)) + \beta(t) - r(t)] + (k_1 q(t) + k_2 (1 - q(t)) \\
+ \beta(t) - g(\beta(t)) - \frac{1}{2} (m_1^2 + m_2^2 + m_3^2) \} dt + \tilde{M}_t,
\]

(4.7)

where

\[
\tilde{M}_t = \int_0^T m_1 dB(t) - \int_0^T m_2 dW_2 - \int_0^T m_3 dW_3.
\]

\(\tilde{M}_t\) is a martingale. Let

\[
\tilde{J}(q, \pi) = \pi(t) [k_1 q(t) + k_2 (1 - q(t)) + \beta(t) - r(t)] + (k_1 q(t) + k_2 (1 - q(t)) \\
+ \beta(t) - g(\beta(t)) - \frac{1}{2} (m_1^2 + m_2^2 + m_3^2).
\]

(4.8)

Our original problem to maximize \(J(x, \pi)\) in (3.1) is equivalent to the following one based on nonlinear filtering technique.

Maximize \(\mathbb{E}_q \tilde{J}(q, \tilde{\pi})\)

\[
\begin{align*}
\text{s.t.} & \\
& dq(t) = [\lambda_1 q(t) + \lambda_2 (1 - q(t))] dt + \frac{k_1 - k_2}{\sigma_1} q(t) (1 - q(t)) dB(t), \ 0 \leq q \leq 1. \\
& q(0) = q_0.
\end{align*}
\]

(4.9)

A strategy \(\tilde{\pi}(\cdot) = \{\tilde{\pi}(t) : t \geq 0\}\) is progressively measurable with respect to \(\sigma\{B(s) : s \leq t\}\) is called an admissible strategy (or admissible control). A Borel measurable function \(\tilde{\pi}(q)\)
is an admissible feedback strategy or feedback control, \( \tilde{\pi}(\cdot) \) is Lipschitz continuous and \( \tilde{\pi}(t) = \tilde{\pi}(q(t)) \) is admissible for each \( t \geq 0 \). That is, the optimal debt ratio is a function of conditional probability. In light of (4.9), we need to find relationship between the optimal debt ratio and conditional probability. The conditional probability can be observed directly from the real data.

Then the value function is defined as

\[
\tilde{V}(q) := \sup_{\tilde{\pi} \in \tilde{A}} \tilde{J}(q, \tilde{\pi}). \tag{4.10}
\]

\( \tilde{A} \) denotes the collection of all admissible strategies or admissible controls. By applying the dynamic programming principle, the value function (4.10) satisfies the HJB equation

\[
[-\lambda_1 q + \lambda_2 (1 - q)] \tilde{V}_q + \frac{1}{2} \left( \frac{k_1 - k_2}{\sigma_1} \right)^2 q^2 (1 - q)^2 \tilde{V}_{qq} = 0, \tag{4.11}
\]

Denote by \( \tilde{V}(q) \) the solution to (4.11). There exists \( C_1 \) and \( C_2 \) such that

\[
\tilde{V}(q) = C_1 \int_0^q \frac{dx}{\exp \left[ \int_0^x R(u) du \right]} + C_2, \tag{4.12}
\]

where

\[
R(u) = \frac{-\lambda_1 u + \lambda_2 (1 - u)}{\frac{1}{2} \left( \frac{k_1 - k_2}{\sigma_1} \right)^2 u^2 (1 - u)^2}.
\]

\( \tilde{V}(q) \) can be obtained by combining with the boundary conditions.

In terms of boundary conditions, recalled that \( q \) is the conditional probability of states of Markov jump process, and \( 0 \leq q \leq 1 \). We need to verify the boundary conditions when \( q = 0, 1 \). When \( q = 1 \), the market is in the bull market regime. The drift of the asset price
is positive and $\mu(t) = k_1$. Then we have

$$
\frac{\partial}{\partial \tilde{\pi}} \left\{ (k_1 + \beta(t) - g(\beta(t))) - \frac{1}{2} \left[ (\sigma_1 - \rho_2 \sigma_3)^2 + \frac{\sigma_3^2 (\rho_3 - \rho_1 \rho_2)^2}{1 - \rho_1^2} + \left( 1 - \rho_2^2 - \frac{(\rho_3 - \rho_1 \rho_2)^2}{\sqrt{1 - \rho_1^2}} \right) \right] \right\} \\
+ \tilde{\pi} \left[ (k_1 + \beta(t) - r(t)) - (\sigma_1 - \rho_1 \sigma_2)(\sigma_1 - \rho_2 \sigma_3) - \sigma_2 \sigma_3 (\rho_3 - \rho_1 \rho_2) \right] \\
- \frac{1}{2} \tilde{\pi}^2 \left[ (\sigma_1 - \rho_1 \sigma_2)^2 + \sigma_2^2 (1 - \rho_1^2) \right] \bigg|_{\tilde{\pi} = \tilde{\pi}_1^*} = 0,
$$

(4.13)

where $\tilde{\pi}_1^*$ is the optimal control when $q = 1$ and follows

$$
\tilde{\pi}_1^* = \frac{(k_1 + \beta(t) - r(t)) - (\sigma_1 - \rho_1 \sigma_2)(\sigma_1 - \rho_2 \sigma_3) - \sigma_2 \sigma_3 (\rho_3 - \rho_1 \rho_2)}{(\sigma_1 - \rho_1 \sigma_2)^2 + \sigma_2^2 (1 - \rho_1^2)}.
$$

(4.14)

When $q = 0$, the market is in the bear market regime at current time. The drift of the asset price is negative and $\mu(t) = k_2$. Then we have

$$
\frac{\partial}{\partial \tilde{\pi}} \left\{ (k_2 + \beta(t) - g(\beta(t))) - \frac{1}{2} \left[ (\sigma_1 - \rho_2 \sigma_3)^2 + \frac{\sigma_3^2 (\rho_3 - \rho_1 \rho_2)^2}{1 - \rho_1^2} + \left( 1 - \rho_2^2 - \frac{(\rho_3 - \rho_1 \rho_2)^2}{\sqrt{1 - \rho_1^2}} \right) \right] \right\} \\
+ \tilde{\pi} \left[ (k_2 + \beta(t) - r(t)) - (\sigma_1 - \rho_1 \sigma_2)(\sigma_1 - \rho_2 \sigma_3) - \sigma_2 \sigma_3 (\rho_3 - \rho_1 \rho_2) \right] \\
- \frac{1}{2} \tilde{\pi}^2 \left[ (\sigma_1 - \rho_1 \sigma_2)^2 + \sigma_2^2 (1 - \rho_1^2) \right] \bigg|_{\tilde{\pi} = \tilde{\pi}_0^*} = 0,
$$

(4.15)

where $\tilde{\pi}_0^*$ is the optimal control when $q = 0$ and follows

$$
\tilde{\pi}_0^* = \frac{(k_2 + \beta(t) - r(t)) - (\sigma_1 - \rho_1 \sigma_2)(\sigma_1 - \rho_2 \sigma_3) - \sigma_2 \sigma_3 (\rho_3 - \rho_1 \rho_2)}{(\sigma_1 - \rho_1 \sigma_2)^2 + \sigma_2^2 (1 - \rho_1^2)}.
$$

(4.16)

We thus derive the representative optimal control debt ratios for respective market regimes specified as (4.14) and (4.16). In terms of estimation, let

$$
G(q) = \int_0^q dx \exp \left[ \int_0^x R(u) du \right].
$$
Then,
\[ C_1 = \frac{\tilde{J}(q, \tilde{\pi}^*_1) - \tilde{J}(q, \tilde{\pi}^*_0)}{G(1)}, \quad C_2 = \tilde{J}(q, \tilde{\pi}^*_0). \]  
(4.17)

The value function \( \tilde{V}(q) \) follows
\[ \tilde{V}(q) = \frac{\tilde{J}(q, \tilde{\pi}^*_1) - \tilde{J}(q, \tilde{\pi}^*_0)}{G(1)} \int_0^q dx \exp \left[ \int_0^x R(u) du \right] + \tilde{J}(q, \tilde{\pi}^*_0). \]  
(4.18)

Recalled that \( \tilde{V}(q) \) is the maximum of \( \tilde{J}(q, \tilde{\pi}) \), then the maximal growth of net wealth is obtained in the case of Markov jump process. \( C_1 \) and \( C_2 \) can be determined as we combine the value functions in the two market regimes and (4.18).

However, due to the complexity and unobservability of \( q(t) \), it is very hard to obtain the explicit solution of optimal debt ratio. Alternatively, we can estimate the \( q(t) \) with the expected fraction of time in bull market denoted by \( \bar{\tilde{q}}(t) \). Recalled that \( \lambda_1 \) and \( \lambda_2 \) are the parameters of Makov chain’s generator defined in (4.1) and (4.2). We have
\[ \bar{\tilde{q}}(t) = \lim_{t \to \infty} \mathbb{E}(q(t)) = \frac{1/\lambda_1}{1/\lambda_1 + 1/\lambda_2} = \frac{\lambda_2}{\lambda_1 + \lambda_2}. \]  
(4.19)

Then we can use \( \bar{\tilde{q}}(t) \) to estimate the optimal debt ratio in (3.5). The estimated optimal debt ratio \( \bar{\pi}^* \) follows
\[ \bar{\pi}^* = \frac{\frac{\lambda_1 \lambda_2 + k_2 \lambda_1}{\lambda_1 + \lambda_2} + \beta(t) - r(t) - (\sigma_1 - \rho_1 \sigma_2)(\sigma_1 - \rho_2 \sigma_3) - \sigma_2 \sigma_3 (\rho_3 - \rho_1 \rho_2)}{(\sigma_1 - \rho_1 \sigma_2)^2 + \sigma_2^2 (1 - \rho_1^2)}. \]  
(4.20)

The estimated time-varying optimal debt ratio can be estimated by expression (4.20) accordingly.

In brief, we have derived four expressions for calculating optimal debt ratio: (3.5), (4.14), (4.15), and (4.20). Expression (3.5) gives the outcome simply based on HJB. Further, (4.14) and (4.15) consider respective market regimes. (4.20) is the general form of (4.14) and (4.15).
because (4.20) is the expected value of (4.14) and (4.15), weighted by their corresponding frequency parameters \( \lambda_1 \) and \( \lambda_2 \). The only difference among the four expressions lies in the first term in the numerator. Initial level of asset return makes a difference. In comparison of (4.14), and (4.15), the higher drift rate of the regime \( (k_1 > k_2) \) give the higher level of optimal debt ratio. \( k_1 \) and \( k_2 \) are contingent on asset return level. Accordingly, the higher asset return level in bull market gives the higher optimal debt ratio.

We further examine the respective partial effect of the included variables in (4.20) by taking their first-order derivatives. For simplicity, let

\[
L = \frac{k_1 \lambda_2 + k_2 \lambda_1}{\lambda_1 + \lambda_2} + \beta - r - \sigma_1^2 + \rho_1 \sigma_1 \sigma_2 + \rho_2 \sigma_1 \sigma_3 - \rho_3 \sigma_2 \sigma_3,
\]

\[
D = \sigma_1^2 - 2\rho_1 \sigma_1 \sigma_2 + \sigma_2^2.
\]

We have

\[
\frac{\partial \bar{\pi}^*}{\partial \lambda_1} = \frac{(k_2 - k_1)\lambda_2}{D(\lambda_1 + \lambda_2)^2},
\] (4.21)

\[
\frac{\partial \bar{\pi}^*}{\partial \lambda_2} = \frac{(k_1 - k_2)\lambda_1}{D(\lambda_1 + \lambda_2)^2}.
\] (4.22)

\[
\frac{\partial \bar{\pi}^*}{\partial k_1} = \frac{\lambda_2}{D(\lambda_1 + \lambda_2)},
\] (4.23)

\[
\frac{\partial \bar{\pi}^*}{\partial k_2} = \frac{\lambda_1}{D(\lambda_1 + \lambda_2)},
\] (4.24)

\[
\frac{\partial \bar{\pi}^*}{\partial \beta} = \frac{1}{D},
\] (4.25)

\[
\frac{\partial \bar{\pi}^*}{\partial r} = -\frac{1}{D},
\] (4.26)

\[
\frac{\partial \bar{\pi}^*}{\partial \sigma_1} = \frac{-2\sigma_1 + \rho_1 \sigma_2 + \rho_2 \sigma_3}{D} - \frac{2L(\sigma_1 - \rho_1 \sigma_2)}{D^2},
\] (4.27)
\[ \frac{\partial \bar{\pi}^*}{\partial \sigma_2} = \frac{\rho_1 \sigma_1 - \rho_3 \sigma_3}{D} - \frac{2L(\sigma_2 - \rho_1 \sigma_1)}{D^2} , \]  
(4.28)

\[ \frac{\partial \bar{\pi}^*}{\partial \sigma_3} = \frac{\rho_2 \sigma_1 - \rho_3 \sigma_2}{D} , \]  
(4.29)

\[ \frac{\partial \bar{\pi}^*}{\partial \rho_1} = \frac{\sigma_1 \sigma_2(D - 2L)}{D^2} , \]  
(4.30)

\[ \frac{\partial \bar{\pi}^*}{\partial \rho_2} = \sigma_1 \sigma_3 , \]  
(4.31)

\[ \frac{\partial \bar{\pi}^*}{\partial \rho_3} = -\frac{\sigma_2 \sigma_3}{D} . \]  
(4.32)

Productivity ratio can determine the affordability of raising debt. The higher the beta, the higher optimal debt ratio (4.25). The higher the interest rate \( r \), the heavier the debt burden and lower the optimal debt ratio accordingly (4.26). The asset return levels in the respective market regimes \( (k_1 \ and \ k_2) \) give positive effects on determining the optimal debt ratio. Both (4.23) and (4.24) show positive outcomes. The frequencies in the Markov chain jump exercise their individual effect too. \( \lambda_1 \ and \ \lambda_2 \) give negative and positive effect, respectively. That is, the frequency in the bull market is likely to lower the optimal debt ratio (4.21) and it is different picture in bear market (4.22). However, the volatility of asset return, interest rate, and consumption ratio do not have significant effect on optimal debt ratio ((4.27), (4.28), and (4.29)). Accordingly we can discuss their respective effects on the optimal debt ratio with respect to their trend, not variance level. The correlations among the asset return, interest rate, and consumption ratio give mixed outcomes. While the effect by \( \rho_1 \) (the correlation between asset return and interest rate) is undetermined, \( \rho_2 \) (the correlation between asset return and consumption ratio) and \( \rho_3 \) (the correlation between consumption ratio and interest rate) demonstrate positive and negative signs, respectively. That is, the optimal debt ratio is likely to increase due to the effect from demand since \( \rho_1 \ and \ \rho_2 \) are all positive. Yet the ratio will decrease in terms of the correlation between consumption ratio and interest rate. In short, the optimal debt ratio is more dependent...
on the levels, not the volatility, of the asset return, productivity ratio, and interest rate. The correlation among the three variables has specific effects. Consumption ratio can be evaluated as the key factor and its respective correlation between asset return and interest rate show specific effect on determining optimal debt ratio.

5 Empirical Analysis

The core components to estimate the optimal debt ratio include productivity rate of capital $\beta(t)$, consumption ratio $c(t)$, interest rate, and asset return $r(t)$, as specified in (4.14), (4.16), and (4.20). All the data series except HPI4Q are downloaded from the Economic Research, Federal Reserve of St. Louis (http://research.stlouisfed.org/). Quarterly data period extends from the first quarter of 1991 to the second quarter of 2013. I plot and discuss the individual series from two perspectives: private and public. Following (4.27)-(4.29), we discuss their individual effect on the optimal debt ratio based on the trend, not variance level.

There are two kinds of productivity rate of capital employed in this study: real disposable personal income change rate for the private level, and GDP growth rate for the public level. Their time series plots are summarized as Figure 1 and Figure 2, respectively. We thus exposit this study from two perspectives: the public and the private. Productivity ratio indicates its positive relationship with affordability of raising debt, as indicated in (4.25). The real disposable personal income change rate series shows a decreasing trend after 2000, especially after the crisis in 2007. GDP growth rate demonstrates an increasing trend after 2000 but a trough around the crisis in 2007. In short, GDP growth rate after 2000 does not fully reflect the growth rate of the real disposable personal income.

The interest rate series adopts Immediate Rates (or Prime Rates) for both the public
and the private sectors. The time series plot of interest rate Figure 3 indicates that the level is lowered for several years since 2000, and reverted back to higher levels until 2007. The reduced interest rate has inflammable effect on the exuberant real estate market and over-indebted households’ leverage, as shown in (4.26). After the crisis in 2007, the level keeps dropping and makes spending attractive until 2009, and remains flat afterwards. These evidences indicate that US government is suspected of manipulating its interest rate level as a policy instrument to fine tune the economy.

In terms of consumption ratios, personal consumption ratio is for the household (private) debt level (Figure 4), and private consumption ratio for the public debt level (Figure 5). Personal consumption ratio is calculated as the ratio of personal consumption expenditure divided by disposable personal income. This ratio keeps its increasing trend until the crisis in 2007. This ratio levels down afterwards. Private consumption ratio equals the ratio of aggregate private consumption over GDP. This series shows a significant increasing trend after 2000, and remains flat between 2003 and 2007. This series picks up its increasing trend in the following periods. In general, private consumption ratio shows a significant long-run increasing trend, even after the crisis in 2007. This incremental curve does not show in personal consumption ratio series. The difference between these two consumption ratios suggests that the increase in private private consumption is not supported by the change in personal consumption level but by various types of leverage.

As indicated by (4.14) and (4.16), the optimal debt ratios vary across market regimes. I adopt the piecewise linear segmentation of the time series of return series to differentiate the bull and bear market regimes. This algorithm is basically used to find the slope of the points of the series via least squares regression at a fixed-length moving window, sliding from the leftmost to its rightmost point of the time series. Along the sliding process, the new slope is compared to the old slope. If the change in slope exceeds the pre-specified angle tolerance, a change-point is recorded as the rightmost point of the previous iteration’s window. If the
change in slope does not exceed the specified angle tolerance, then the old slope is updated (in a running average sense), and the algorithm continues as usual. Consequently the return series can be differentiated into two regimes \( q = 0 \) for bear market and \( q = 1 \) for bull market) and the respective set of parameters can be estimated accordingly: \( k_0 \) and \( \lambda_0 \), and \( k_1, \lambda_1 \), where \( k \) and \( \lambda \) denote the expected value and frequency of the respective regime.

I choose two kinds of asset return series for empirical analysis: House Price Index and Wilshire 5000 stock index in USA. The former has a higher level than the latter. The former index is representative of the real estate market and indicative of the financial crisis since 2007; the latter index can be regarded as the most comprehensive indicator of the aggregate equity market condition. The House Price Index is downloaded from the Federal Housing Finance Agency (http://www.fhfa.gov/) and the series represents the change rate over previous four quarters of US housing prices. This series is denoted as HPI4Q. Wilshire 5000 stock index, denoted as Wilshire5000, is a market-capitalization-weighted index of the market value of all stocks actively traded in the United States. Wilshire5000 gives the most comprehensive measure of aggregate stock market performance. Wilshire5000 can be regarded as one of the representative indicators of the overall economic activities. Accordingly, the productivity rate can be treated as the growth rate of HPI4Q and gross domestic production (GDP), respectively. There are two kinds of measures for consumption ratio: (1) The personal disposable income level for the private level, following Stein (2012), or (2) The aggregate private level with respect to GDP level for the public sector. Accordingly we can set up four scenarios for empirical analyses with respect to two asset return series and two consumption ratios, with the same interest rate series. Two kinds of actual debt ratios are included to contrast with the estimated optimal debt ratio for all regimes (ODR-All): Household Debt Service Payments as a Percent of Disposable Personal Income (TDSP) and Household Financial Obligations as a percent of Disposable Personal Income (FODSP).

\(^2\)I do not consider the aggregate bond index because government bond is included in the index, and significant correlation between both and endogeneity issue should be avoided.

\(^3\)ORD-All estimates lie between the optimal ratios estimated under the bull and bear market regimes.
FODSP is a broader term of debt ratio than TDSP. 

We first select the asset rate of return series as HPI4Q series, and the plot of piecewise linear segmentation is presented as Figure 10. The respective set of parameters of the expected value and frequency for the bull and bear market regime are estimated as: $k_0 = 0.0335$ and $\lambda_0 = 0.011818$, and $k_1 = 0.0282$, $\lambda_1 = 0.11528$. If we focus on the household debt level, the plot of ODR-All, TDSP, FODSP is demonstrated as Figure 6. In most of the quarters, ODR-All is significantly lower than those levels of TDSP and FODSP. It indicates that US households generally allocate a higher portion of income for debt payment than the level specified by ORD-All. If we target on the public debt level, we differentiate the debt ratios for household, Federal, State and local governments, and the total public debt. These ratios are denoted as RHousehold, RFederal, RState&Federal, and RTOTAL, respectively. RTOTAL is the summation of RHousehold, RFederal and RState&Federal. They are summarized as Figure 7. While the Federal government lowers its debt ratio since 2000, the household debt ratio shows a significant increasing trend. Noteworthy, ORD-All does not show the corresponding uptrend but a downtrend instead. That is, ORD-All reflects the economic downturn and the drop in productivity rate of capital. The increasing household debt ratio during that period is not sustained by the actual economic performance, as indicated by productivity rate. Further, the estimated ORD-All shows a significant drop in the level after the 2007 crisis, the household debt level also gives a significant downturn. That is, the estimated optimal debt ratio reflects the abrupt economic downturn. The downswings of household debt ratio indicates that indebted US households cannot afford to spend the way they used to and they have to pay down the debts they ran up in the bubble years. Meanwhile, the public debt ratios surge. The State and local governments remain their debt ratio level. Yet the Federal government soars its debt ratio since then, so

\[4\] TDSP and FODSP are two approximate estimates of household debt ratios. They are the ratios of total required household debt payments to total disposable income. The limitations of current sources of data make the calculation of the ratio especially difficult. See http://www.federalreserve.gov/releases/housedebt/about.htm
did the total public debt level accordingly. The Federal government steps in and initiates series of QE to stimulate the economy in a massive way by issuing more public debt into the market, so as to substitute the chilling private expenditure. It is an urgent rescue policy, though productivity ratio does not afford such massive scale. The total public debt ratio shows a significant increasing trend after 2000, and a soaring curve after 2007 almost up to 170% of GDP. Yet this debt overhang is not echoed by the actual economic rebounce, at least reflected in the shrinking private debt ratio. It can be due to that this policy measure reduces economic activities as the households debt ratio is crowded out or the households slash their spending. ORD-All was boosted for several quarters after the economic stimulus but the level remains sluggish afterwards. The effect of economic stimulus can be conservative and short-lived. The over-indebted US governments, especially the Federal government, is loaded with even heavier leverage.

I further choose Wilshire5000 as the asset return series and the plot of piecewise linear segmentation is presented as Figure 11. The respective set of parameters of the expected value and frequency are estimated as: $k_0 = 0.0234$ and $\lambda_0 = 0.11667$, and $k_1 = 0.0310$, $\lambda_1 = 0.10761$. We first look at the household level and the plot of ODR-All, TDSP, and FODSP is summarized as Figure 6. The ODR-All estimated with Wilshire5000 is lower than those estimated with HPI4Q. US real estate market gives a higher return rate than the stock market, reflecting in ODR-All estimates accordingly. The actual household debt levels are significantly higher than the estimated ODR-All. In aggregate, US households are lavish in leveraging and spending, more than the specified level of ODR-All. If we turn to the public debt level, Figure 9 shows ORD-All is much lower than those estimated with HPI4Q. The similar situation also happens at the household level. In brief, the US household and governments have spent more than the stock market performance can afford. While the households reduce their debt ratio, the public sector hikes its debt ratio to stimulate the economy but the effect is below their policy expectation.
6 Conclusion

We develop a SOC model to derive the formulae of the optimal debt ratios for the public and the private sectors at two different market regimes. This refined SOC model is based on Bellman’s technique to solve the dynamic programming issue. A hidden Markov chain model is included to model discrete market movements. The nonlinear filtering technique is introduced for the estimation purpose. This model considers productivity of capital, asset return, interest rate, and market regime. The estimated optimal debt ratio is reflective of actual economic conditions. The empirical analyses based on both the US markets of real estate and equity indicate that the actual debt ratios are significantly higher than the estimated levels in both the public and the private sectors. The launched series of QE hardly deliver the expected policy effects to revitalize the economy.
References


Figure 1: Time Series Plot of Real Disposable Personal Income

Figure 2: Time Series Plot of GDP Growth Rate
Figure 3: Time Series Plot of Interest Rate

Figure 4: Time Series Plot of Personal Consumption Ratio
Figure 5: Time Series Plot of Private Consumption Ratio

Figure 6: Time Series Plot of ODR-All, TDSP, FODSP (HPI4Q, personal)
Figure 7: Time Series Plot of ORD-All, RHousehold, RFederal, RState&Local, and RTOTAL (HPI4Q, public)

Figure 8: Time Series Plot of ODR-All, RHousehold, RFederal (Wilshire, public)
Figure 9: Time Series Plot of ODR-All, TDSP, FODSP (Wilshire, personal)

Figure 10: Piecewise Linear segmentation (HPI4Q)
Figure 11: Piecewise Linear segmentation (Wilshire)